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MODELLING THE VAULT OF SAN CARLO ALLE QUATTRO FONTANE

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Abstract. San Carlo alle Quattro Fontane, as in general roman baroque architecture, has a remarkable complexity in his forms and volumes articulation, not easy to control trough a linear relief. In the present analysis, mathematical representation of planar curves have been superimposed to picture and drawings leading to a three-dimensional parametric model of the vault. The use of Mathematica software, combining graphical interface to powerful numerical and symbolic evaluation, has allowed construction and visualization of complex models with a measure of their validity. We show that the curve at the basis of the vault, an oval formed by circular arcs very close to an ellipse, is better approximated by an elongated Epitrochoid and the position of the lacunars can be related to simple hypothesis on their shape and geometry. Similar problems of modeling other vault's decorations can be analyzed by changing parameters.

Key words. Roman baroque architecture, surfaces and curves, parametric equations, Mathematica software, three-dimensional models.

1 Bi-dimensional Analysis

Borromini's architectures, despite its appearance, are usually based on an elementary geometry at the beginning of the design process (triangles and circles), that become progressively more complex during the evolution of the drawing defining spaces and shapes, attempting a typical baroque conception. In this case for example the vault's geometrical origin is based on an apposition of two equilateral and specular triangles. It is possible to say that many interpretation of the geometrical construction of the vault's oval can be reconnected to the ellipse formula as in the following drawing by Paolo Portoghesi. They are mainly composed by an empirical union of different circular arcs centred at vertices or at the barycentre of the equilateral triangles.

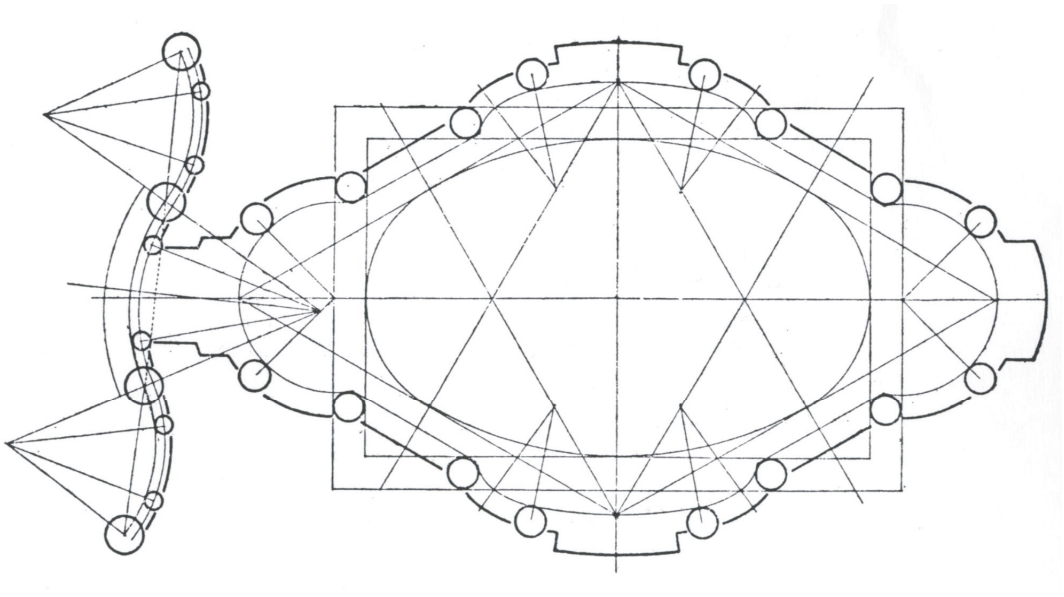

Figure 1: A scheme of the geometrical construction of the oval (P. Portoghesi)

1.1 Description of the bi-dimensional curve

Trying to define architectural forms in mathematical models, one of the most common curves that can be used is the epicycloid: it is the trajectory of a given point on a circle of radius r which rolls around a fixed circle of radius R.

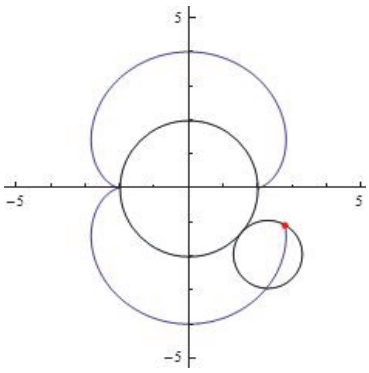

Figure 2: An epicycloid with $R = 2\,r$

The equation of the epicycloid is:

$$\begin{aligned} x(t) &= (R+r)\cos t - r\cos\left(\frac{R+r}{r}t\right) \\ y(t) &= (R+r)\sin t - r\sin\left(\frac{R+r}{r}t\right) \end{aligned} \tag{1.1}$$

As the ellipse is a particular case of an epicycloid (an ipotrochoid, with the chosen point fixed at distance $h > r$ from the center of the rolling circle), we tested several 2-cusped epicycloid (nephroid). Both ipotrochoid and epitrochoid (with $h < r$) eliminates the cusps but the one that better describe the vault is the epitrochoid.
In Fig.3 we compare the shape of different epitrochoids together with the corresponding graph ot their curvature *K(t);* when $h = 1/3$ two features are reached: regions with almost constant curvature (circular arcs) and points with zero curvature (linear neighborhood). Moreover such a curve is as regular as the ellipse and much more regular than an oval made of several circular arcs.

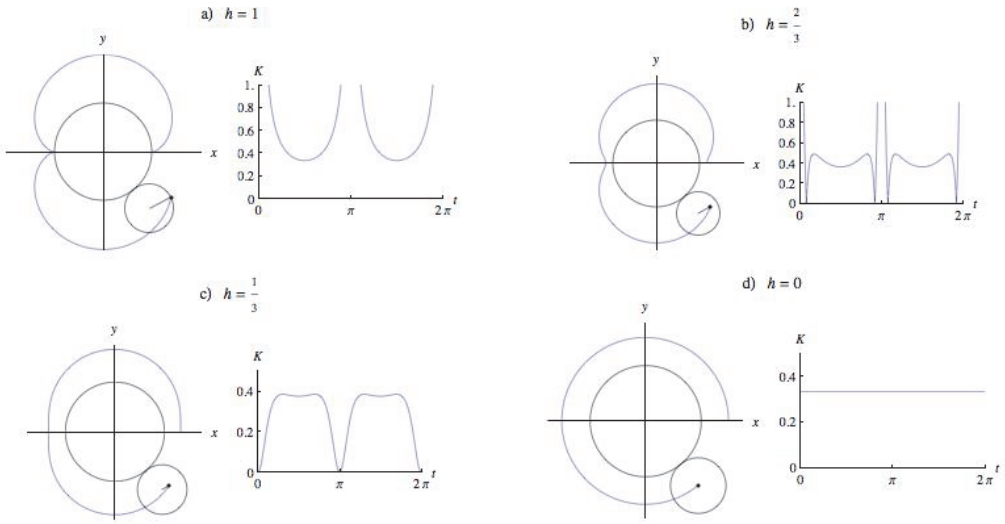

Figure 3: *Epitrochoids whith h = 1, 2/3, 1/3 and 0.*

Some additional parameters *c*, *d*, were introduced, to stretch the curve along its axis, for a better adaptation to the vault's shape:

$$
\begin{aligned}
x(t) &= c\left(R+r\right)\cos t - h\cos\left(\frac{R+r}{r}t\right) \\
y(t) &= c\left(R+r\right)\sin t - h\sin\left(\frac{R+r}{r}t\right)
\end{aligned}
\tag{1.2}
$$

As it is possible to see in Fig:4, the ellipse (red) doesn't follow the vault shape along the diagonals, and in the middle is very close to a circular arc (green). This particular epitrochoid actually in the middle of the vault has curvature very close to 0, since it is quite similar to a line, whether in its lateral sides is very close to a circle.

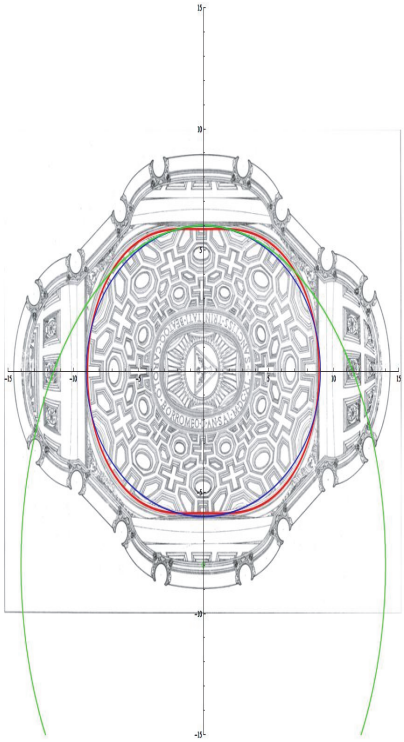

Figure 4: *Comparison between an ellipse (blue), a circular arc of an oval (green) and the epitrochoid (red) with R = 2, r = 1, c = 2, d = 1.5 and h = -1/3.*

1.2 Variations of some parameters

It has been quite interesting to notice that this model could be adapted to some other architectonical solutions in baroque architecture. Actually Borromini's influence has been very important for some other counter-reformists country during the second half of the seventeenth century, as for example Bavaria and also Bohemia. Some churches in Bavaria as Asamkirche in Munich designed by Asam Brothers presents a very similar geometric shape based on 2- oval which is exasperate the S. Carlo shape apparently similar to an ellipse. Another building is the Chapel of Epiphany in Smiřice by Krystof Dientzenhofer which represent in his plan and in his geometrical construction a clear reference to S. Carlino and it determinates how his geometry does not have his origin in an ellipse but in empirical shapes made of the apposition of simple geometrical elements.

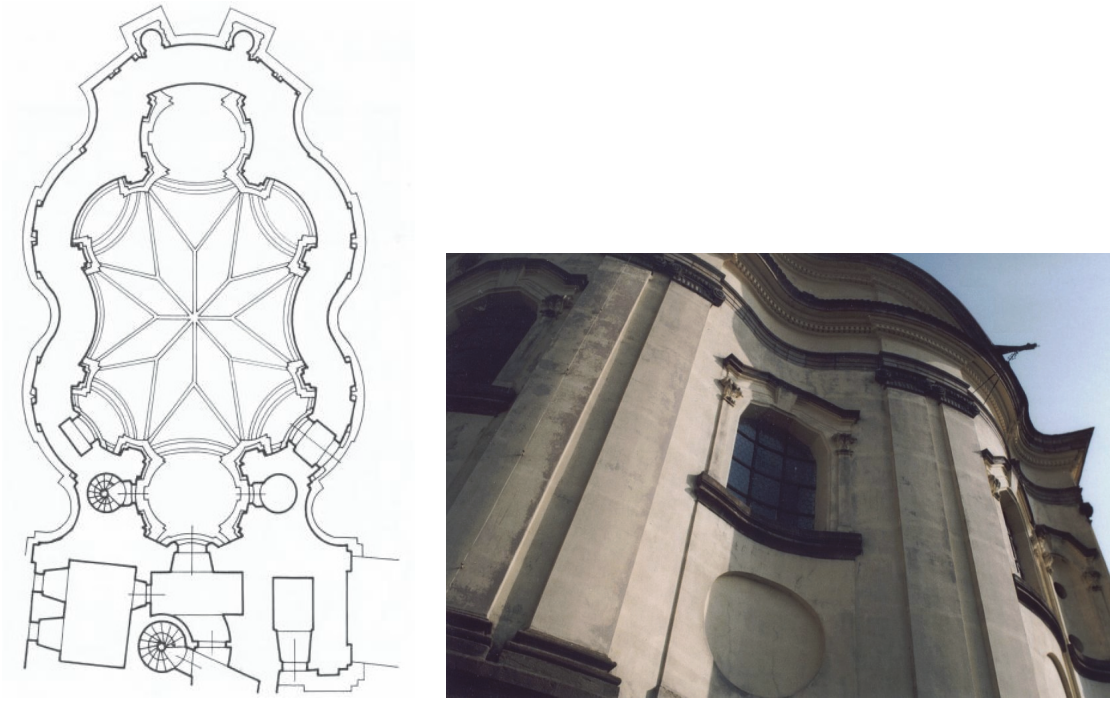

Figure 4: *Examples of influence of roman baroque architecture.*

There could have been many other references especially in later-baroque bohemian architecture but these two could be considered the most explicit in relation to the work we have done.
Here are some other examples:

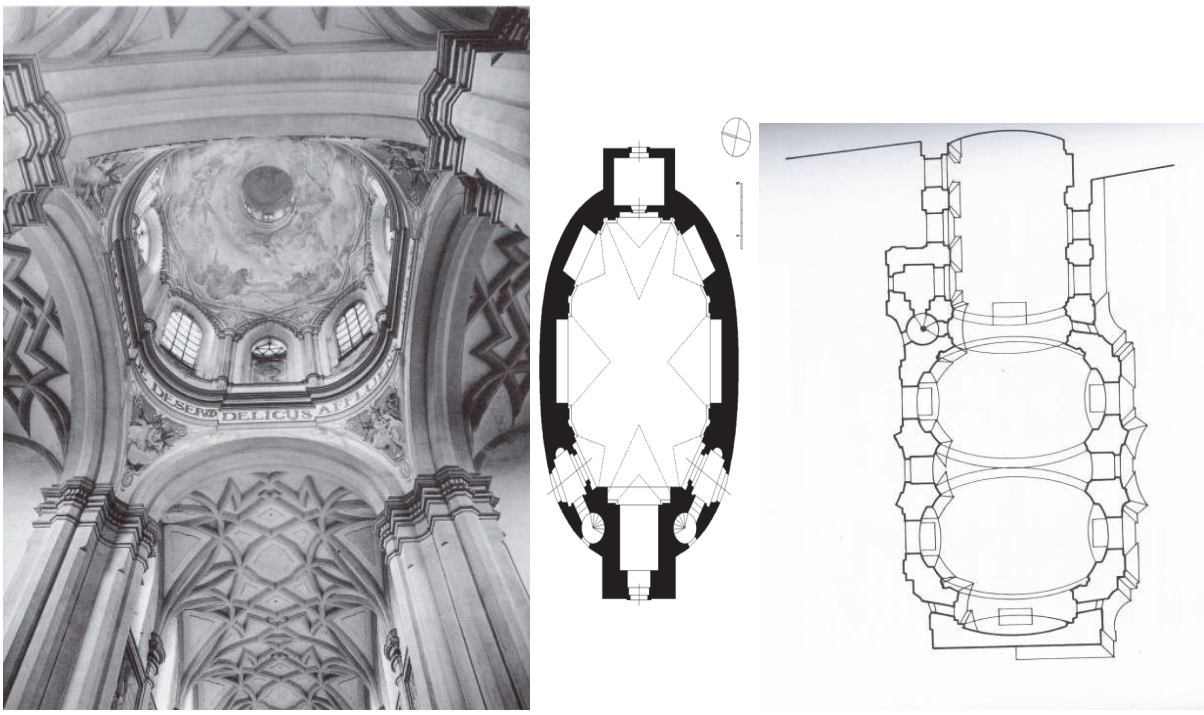

Figure 4: *Other examples ...*

2 Three-dimensional model

Starting from the basis of the vault (the stretched epitrochoid with *R=2, r=1, c=2, d=1.5, h=-1/3)* a simple model of the surface can be contructed assuming an ellipsoidal shape: in parametric coordinates can be formulated as:

$$\gamma(u,v)=\left(6\cos u\,.\frac{1}{3}\cos 3u\,.\sin v,\frac{9}{2}\sin u\,.\frac{1}{3}\sin 3u\sin v,\frac{25}{6}\cos v\right) \tag{1.2}$$

The lacunars seems to be bounded by coordinates curves ("parallels" and "meridians") in a very regular pattern: alternating crosses and regular octagons on increasing levels seems to rescale towards the top of the vault. Given the nth vertical level and the number N of lacunars for any level we can define position and size of a tassel of the surface by the spherical coordinates of its centre (u_0,v_0) and the angular "radius" in horizontal (u_r) and vertical (v_r) direction:

$$u_0=\frac{2\pi}{N} \tag{2.1}$$

$$v_0^n=\frac{2\pi}{N}2b^n\left(1+b\right) \tag{2.2}$$

$$u_r^n=u_r^0=\frac{\pi}{N} \tag{2.3}$$

$$v_r^n=\frac{\pi}{4}b^n\left(1-b\right) \tag{2.4}$$

where b is the ratio between vertical radii of two successive tassels, that is

$$b=\frac{v_r^{n+1}}{v_r^n} \tag{2.5}$$

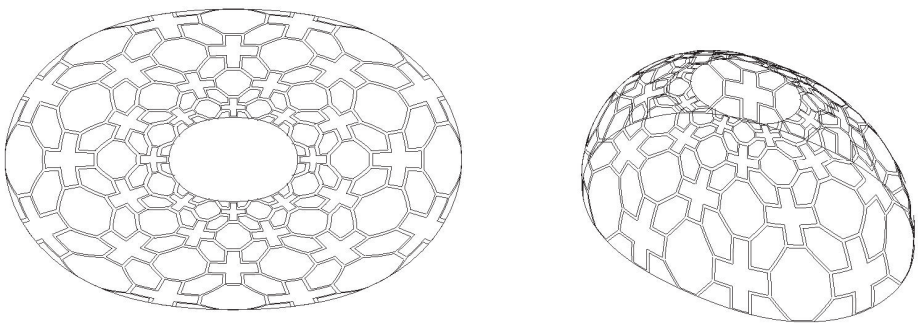

Figure 5: A model of S.Carlino's vault.

In order to fix the value of b and of the vertical size of the first tassel we make the hypothesis that it would be possible to fit an infinite number of successive tassels of constant ratio which ends up at the pole and which looks like squares with some accuracy.

The first hypothesis implies that the sum of the vertical size of the infinite tassels in angular coordinates would be exactly $\pi / 2$, which gives the condition:

$$\frac{\pi}{2} = \sum_{k=0}^{\infty} 2 v_r^k = \sum_{k=0}^{\infty} 2 v_r^0 b^k = 2 v_r^0 \sum_{k=0}^{\infty} b^k = 2 v_r^0 \frac{1}{1-b} \tag{2.6}$$

that is

$$v_r^0 = \frac{\pi}{4}(1-b) \,. \tag{2.7}$$

The condition of tassels of similar size on both vertical and horizontal direction, in a spherical approximation of the surface, can be fixed by the condition (which depends on the vertical position of the tassel)

$$v_r^n = \frac{\pi}{N} \sin v_0^n \tag{2.8}$$

or more explicitly

$$\sin \frac{\pi}{4} b^n (1-b) = \frac{N}{4} b^n (1-b) \,. \tag{2.9}$$

For $b = 2/3$ the condition is well satisfied for tassels after the second level, as shown in Fig.6.

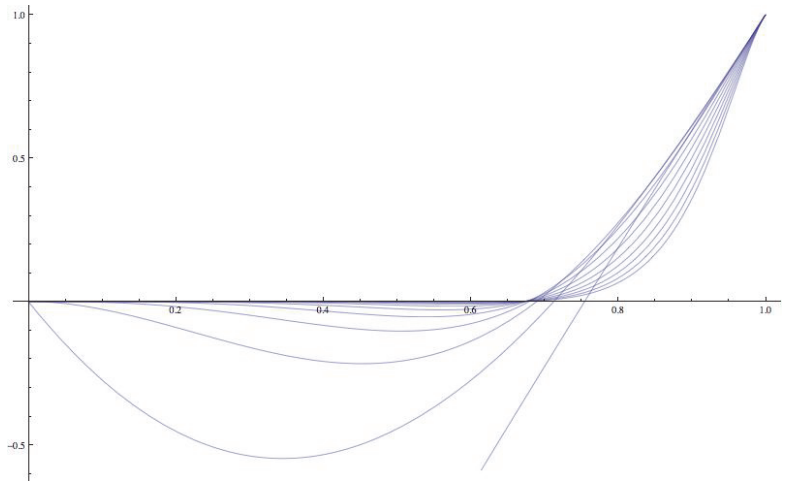

Figure 6: Condition (2.9) on square tassels for different levels versus *b*.

In the model surface the same condition, given as the sides ratio, is satisfied depending on vertical and horizontal position for $b = 2/3$ as shown below for eight horizontal tassels in 4 vertical levels:

Figure 7: The model (b=2/3) superimposed to a picture of S.Carlino's vault.

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