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SIERPINSKY TRIANGLES IN STONE, ON MEDIEVAL FLOORS IN ROME

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Abstract. On the floors of churches in Rome, dating from the 11th century, a particular design can be recognized, much similar to what we today call the Sierpinsky triangle. Such floors are in *opus alexandrinum*, i.e. in pieces of stone of different sizes cut into the desired shape. The question then arises on the elementary shapes, their size and their lay out plan, or composition rules. Multi-scale composition is typical of the floors of the *Marmorari Romani*, loosely known as Cosmati, that naturally point to a fractal analysis. We found that a particular composition is present in several of these floors, more explicitly recognizable as what we today call a Sierpinsky Triangle, i.e. a subdivision on finer and finer scale of self-similar triangles. The composition is either isolated in the floors on red porphyry disc, or weaved into lattices. The instances of Sierpinsky triangles we find are all at least iterated up to three levels.

Key words. Sierpinski Triangle, Multi-scale, Fractal, Cosmati, Marmorari, Medieval art

Mathematics Subject Classification: 01A07, 01A35, 01A60

<<...Ma come mai tali esperienze potrebbero avere un valore probante al di là dei limiti in cui effettivamente furono esperite? La nostra mente supplisce alle esperienze effettuate con esperienze immaginate, la cui possibilità di ripetizione indefinita ci porge la costruzione ideale di una serie infinita di numeri. >>

<<…But, how could such experiences have a cogent value beyond the limits in which they had actually been experienced? Our mind makes up for them with imagined experiences, whose possibility for infinite repetition gives us the ideal construction of and infinite sequence of numbers. >>

Federico Enriques [1]

1. Introduction

In visual arts as well as in musical composition, top-down or down-top procedures distinguish working respectively from the large scale to the small or rather from the small-scale elements assembled to compose the large scale. These are complementary points of view, seen together in the mathematical process of rescaling and studying features at different scales. For instance, "Coarse graining", is a down-top statistical procedure, at the base of the block spin renormalization introduced by Leo Kadanoff in the '60's and the subsequent Nobel prize to Wilson. On the other hand in dynamical systems, when analyzing strange attractors, all characterized by a self-similar structure on fine scales, we rather seek top-down rescaling.

We find recursive subdividing (or top-down) procedures in the mosaic medieval floors of central Italy, in the works internationally known as Cosmati floors, due to a group of families today referred to as *Marmorari Romani* in scholarly literature. The idea of a self-similar organization in Cosmati pavements was already suggested in [2] and we go further in this direction, by documenting explicit appearance of a Sierpinski isolated triangle on the floor of several Romanesque churches, and by introducing the idea of a "self-similar carpet". In this documentation, we make ours the caveat of the Yale group [3], that to make the case for a fractal in an artifact, this should show at least three clear scale levels of iterations. Sufficient iterations assure that, looking at the set, perceptively suggests one of the essential features of fractal sets, i.e. that the limit set of the iteration process be a Cantor set, and not isolated points, as would happen in other rescaling procedures admitting a limit.

Starting with the middle-third Cantor set in 1883 [4], more and more sets have been mathematically defined with a top-down procedure, by subdividing a given set indefinitely in an iterative process; these mathematical objects provided rigorous examples for seemingly very abstract problems as the hypothesis of continuum, and the very concepts of dimension and measure. Sierpinski triangle, or gasket, is one of these seminal sets.

The paper is organized as follows: §2 reviews the concept of Sierpinski triangle, or Sierpinski gasket, its construction and its mathematical properties, including dimension and topological properties. In §3 we review the historical information available today about the *Marmorari Romani*, their geographical realm of activity and we analyze their composition techniques stressing that, besides what has been called a "vocabulary of (their) motives", there exists a "repertoire of compositional rules", which often result in a self-similar structure of a set. In §4 we report the first results of the ongoing documentation of the Sierpinski triangles explicitly present in the floors by the *Marmorari Romani*, as an isolated motive, on a colored marble disk, and we also document the ability of the artisans to deal with curvilinear shapes, much in the same fashion.

2 Sierpinski triangles, their mathematical properties

A Sierpinski triangle is a set obtained as intersection of successive subdivisions of an equilateral triangle. At each step of the recursive procedure, an equilateral triangle of side l is divided into four identical equilateral triangles of side $l/2$; now from this figure the internal triangle is deleted, whose vertices are at the middle points of the original sides. The overall figure results in three identical equilateral triangles, organized around the deleted one. The procedure is iterated over all the triangles remaining at any given level. Thus, we begin with one triangle, and obtain three triangles; at the second level we subdivide each of them again, deleting the inner one, and obtain 9 triangles, and so on, obtaining, at level n, a set S_n consisting of 3^n equilateral triangles. The Sierpinski Triangle S, or Sierpinski Gasket, is the limit set of this procedure, i.e. $S = \bigcap_n S_n$.

A Sierpinski triangle has interesting topological and dimensional properties, which can be readily verified explicitly, due to the recursive definition of S. We refer to [5] for a review of different properties of S, in top-down, and also in down-top procedures as it when arises as an attractor of a dynamical system. We address here the topological and measure properties.
Topologically, S is both a "perfect" and "nowhere dense" set, i.e. a closed set consisting of points that are all accumulation points for a sequence of points of the set, and a set which contains no open (non empty) subsets. Thus, such a set has neither isolated points, nor interior points: it doesn't fill completely, or densely, any part of a space, and its cardinality is that of the continuum. The question arises then as to the measure and the dimensionality of such sets, as a quantification of their property to fill their container space. Measure of a Sierpinski Triangle is zero, as one can easily verify by computing the total area of the equilateral triangles contained at the n^{th} level, and then passing to the limit on n.

A fractal set A is a set with non-integer dimension. The study of "dimension" has seen the efforts of mathematicians throughout the 20^{th} century and addresses how much a lacunous set fills up a portion of its container space. The general definition of dimension of a set can be quite complicated to handle computationally. In recent times, a more computable definition has been introduced. This definition is called the "information dimension" of A. Such definition is particularly manageable when the set is defined by recursively subdividing and deleting. The information dimension, also called the box-counting dimension is computed on especially simple coverings [6, with the accurate bibliography therein]. If B_ε is a ball of radius ε, we can define:

$$N_\varepsilon = \min \{\#(B_\varepsilon)/A \subseteq \cup_\varepsilon B_\varepsilon\},$$

then the dimension of the set involves studying how N_ε grows with ε:

$$\dim (A) := -\lim \ln (N_\varepsilon)/\ln (\varepsilon) \text{ as } \varepsilon \to 0.$$

In the case of the Sierpinski triangle S, we choose $\varepsilon = (\sqrt{3})/3 \cdot 2^{-n}$. At each level, each equilateral triangle is completely covered by the circumscribed circle, so we need 3^n circles of radius $(\sqrt{3})/3 \cdot 2^{-n}$ to cover the S_n set, yielding:

$$\dim(S) = \lim_n -\ln (3^n)/\ln ((\sqrt{3})/3 \cdot 2^{-n}) = \ln 3/\ln 2.$$

Thus the Sierpinski triangle is a fractal set, i.e a set whose dimension is not an integer number.
The self-similarity of Sierpinski Triangle is assured by its method of definition, and is in fact what makes it easy to compute its properties and dimension.

3. The medieval floors of the *Marmorari Romani*

3.1 Mosaic floors in Rome: some history

In Roman tradition, the mosaics were used already in ancient age. The mosaic was used mainly as a protective waterproofing layer, especially in large public places such as baths. As the Romans conquered new populations, they also acquired new culture, new techniques and new ways of building. For instance the art of mosaic changed and took on significance, both functional and decorative, when Rome conquered Greece. Later when quarries from lands of the Roman Empire as North Africa and Syria supplied colored marbles yet unseen, new modes of mosaic were developed, and came to decorate villas and cult places up to the dissolution of the Empire in the 4^{th} century. In the subsequent centuries mosaic saw the intermingling of the techniques and compositional rules of *opus alexandrinum*, with the new materials (including gold) and compositions coming from the refined ritual needs developed in Bisanzium.

In 11^{th} and 13^{th} Century, the general process of renovation of the Church by Pope Gregorio VII (1073-1085), saw the diffusion of new, richer techniques of construction and style of decorations. In the 11^{th} century artisan marble-workers developed new techniques of mosaics for floors and for columns and other vertical architectural elements, at times with Benedictine monks who had traveled east as committents; the families of marble-workers eventually gained the exclusive right to reuse marbles from the imperial ruins, which, as such, belonged to the Pope. Their techniques can be seen to this day in the territories once governed by the Vatican. The artisans are generally known as Cosmati, but beside the Cosmati there were several other Roman families, like the Vassalletto. All them are today referred at by scholars as *Marmorari Romani* [7], the name they later took in 15^{th} century when they gathered in an official guild, operative to this day.

Their mosaic style is based on a composition of tiles, circular elements and bands. The tiles, or *tesserae*, were cut in different geometrical shapes to compose patterns, like in the ancient *opus alexandrinum* (Emperor Alexander Severus, 222-235) and were mainly in white marble, red porphyry and green serpentine. The round elements, or *rotae*, are disks sliced from ancient columns of porphyry or serpentine. In fact it is typical of the Romanesque period the reuse of material from old buildings to build new ones, that often were churches.

The general composition is based on a subdivision of the floor in a central part with a sequence of five or more *rotae* linked by interweaving bands called *guilloche*, and *quinconci* - composed of five *rotae*, at liturgically important sites of the central passage. The two lateral areas are subdivided in rectangular regions, *tappetini,* and filled by a variety of geometrical compositions of colored marble tiles in various scales.

As Kim Williams says in [8] "Cosmati floors weave, with a sense of order, a dialogue at different levels and on different scales: operating the largest division of the interior of the church aisle and rectangular areas side, playing a wide variety of intermediate symmetries (…) and finally, at the local level, leading in some areas of the eye to explore deeper into the surface through the dramatic symmetry of self-similarity".

3.2 Compositional techniques: fragmentation and composition into "carpets".

The *Università dei Marmorari Romani* is still alive, and among their activities is some documentation of instruments and their history [9]. There is no mention, though, of compositional techniques, which clearly subtly distinguish different artisan ateliers. As is well known and documented, the Roman families of *Marmorari* had exclusive right to re-use the marble from Imperial monuments in disuse. This very fact resulted in the literal fragmentation of the stones. Moran-Williams [10], carefully analyze the classification of the motives into tessellation groups, and propose finer ones. We think it is also important to analyze the multi-scale composition: the stones were cut into elementary geometrical shapes, in various scales. Two general geometrical composition rules*, ad quadratum* and *ad triangulum*, are well known in Roman and Medieval art. Both of them have strong symbolical implications that we will not address here, and both are liable for several designs. On an elementary level (i.e. reducing to the simpler element), *ad quadratum* (fig.1) consists of a square, overdrawn with another square, whose vertices are at middle point of its sides. Thus the second, inner, has diagonal the same size as the side of the outer. The overall picture now is composed of a square and four rectangle triangles. One can also say the inner square is rotated by $\pi/4$. If the second square has instead the same size of the first, we obtain an 8-pointed star. Also this is called *ad quadratum*. In any case, some isosceles triangles result in the procedure. The first procedure is the one that naturally points to recursion possibilities, and is in fact exploited at several scales by the *Marmorari*, in the first centuries of their activity. Later, possibly under the influence of southern Italy, other *ad quadratum* appear.

Again an elementary level *ad triangulum* consists of an equilateral triangle overdrawn with another one, whose vertices are at the middle points of its sides. The overall picture is now made of four equilateral triangles; the squares and triangles thus obtained can be reprocessed in one of the two ways, using smaller tiles. Due to the rescaling, all sizes of the triangular and square *tesserae* are in a precise relation, so that an atelier could carry colored *tesserae* already cut to mount a pavement.

Fig. 1 Ad quadratum carpets Santa Maria in Cosmedin, Rome, left and SS. Giovanni e Paolo (Rome), right

Ad quadratum and *ad triangulum* are, per se, rules of subdivision. While it is reported that the *Marmorari* worked by filling [9,11], we think our change in perspective is what can account for the general ability of the *Marmorari Romani* to work controlling different spatial scales: motives where planned as subdivisions, and laid as filling when on premises. In this way, naturally, by successive subdivision of triangles into triangles, textures result, that we will call Sierpinski carpets, at times worked in equilateral triangles, and at times in isosceles right triangles (fig. 2).

Fig. 2 Sierpinski carpets, Santa Maria in Cosmedin, Rome

4. The Sierpinski Triangle in Stone.

Not only one can ascertain the existence of a texture in all the floors by the Marmorari Romani, that coincides with a Sierpinski carpet, but in several cases, we find an isolated Sierpinski triangle (fig. 3), placed on a marble colored *rota*, in red porphyry or green serpentine. In such design it is very clear that smallest white (or yellow) stones are not at all fillings, but the actual structure of what we today we see as the limit set, underlined by being all in one color, all in one size.

Fig. 3 left: San Clemente, Rome (late 11th century) foto Carlini;
right. Santa Maria Maggiore Civita Castellana (12th century) foto Williams.

We present here a photographical documentation of some of its occurrences. We found other instances, for example in the Basilica di Santa Cecilia and in San Crisogono (Rome, Trastevere). In the process of documenting the presence of isolated Sierpinski triangles, we realized a new study is needed, into its occurrence together with other features. For instance, the orientation of the triangle within the church plan is a debated question, especially in the transitional time under study (fig.4).

Fig. 4 San Lorenzo fuori le mura, Rome, (floor 13th century).
Pavement at the base of the major altar. Rectangle is centered on the altar.

The stone triangle is usually on the floor and rather large, but it can also be found in other architectural elements, not meant to be walked upon. In the latter case, the motives are in smaller scale and vitreous matter could be used instead of stone, allowing brighter colors, golden leaf and a general more refined processing (fig. 5). The use of vitreous matter in minute geometrical compositions on the spiral columns is one of the most magnificent accomplishments of this artisanship.

Fig. 5 San Lorenzo fuori le mura, Rome (date unkown) altar. Foto Conversano.

We stress here that we are abiding by the caveat that to claim for a self-similar organization, not only the rescaling should point in the limit to a Cantor set, rather than to isolated points, but also that at least three levels of rescaling should be visible. We therefore include also an example (fig. 6) in which rescaling is clear down to four levels, albeit the largest central triangle has been filled by another Sierpinski. If it had been left void, we would have a five level subdivision.

Fig. 6 SS. Giovanni e Paolo (13th century), Rome

These medieval floors are all in cult places, and the question is open, as far as we know, as to the symbolic content of such explicit self-similar triangle. We know that this triangle has no liturgical symbolism attached[1].

4.1 Curvilinear Sierpinski.

The ability of the *Marmorari* to plan and lay stones at different scales can be all the more appreciated in the motives framed by curvilinear elements. In these motives it is clear that "filling" involves a deep knowledge, as we all know today from the mathematical tools exploited in image processing and compression. Again here, documentation resulted in the need for a separate mathematical study; we present here some examples of motives, which are spread quite widely, in all the churches we visited. We think that these are the difficult ones to obtain and to plan.

Among the Roman marble in ruins, as we said, were the remains of glorious columns of various sizes and different colors: the red of porphyry, the green of serpentine, the yellow of *broccatello di Spagna*. The columns were literally sliced into disks, yielding *rotae* of different sizes, that were then framed and united by a sinusoidal wave of smaller stones. But we also find that either a disk is presented in its entire area, as a larger scale element in the church, or else it was divided in 4-fold or 6-fold rosettes (fig. 7, 8). Therefore the *marmorari* had in their vocabulary not only circles, but also compositions of arcs. They also had the tools to work arcs [12], although we have not found a historic study of the appearance of the different compasses and instrumentation. The rosettes have strong symbolical implications[2], and both types can be composed into a tessellation of the plane, respectively in square and hexagonal lattice. We find that whenever a rosette is displayed, either as such, or into a tessellation (fig. 9), the resulting curvilinear wedges are processed much as in an *ad triangulum*, with the subtlety of subdividing a triangle whose sides are circular arcs, and resulting in a figure which appears self-similar, but not under rescaling according to Euclidean metric.

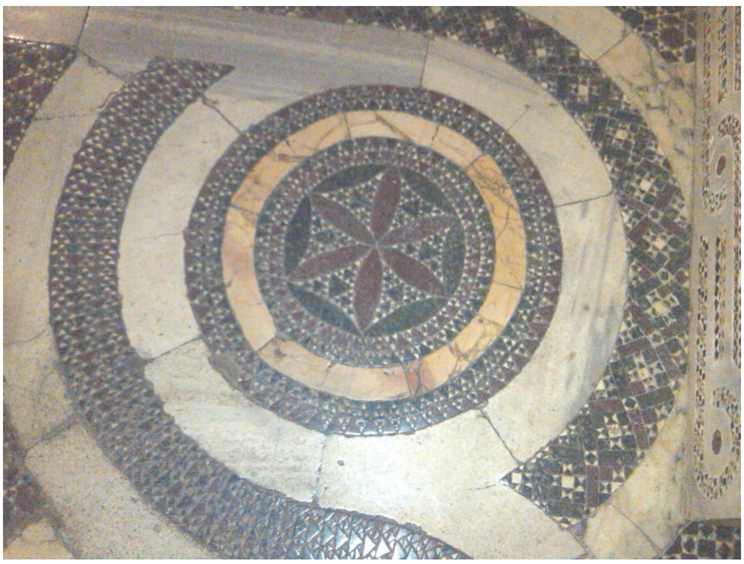



Fig. 7 San Lorenzo fuori le mura (floor 13th century), Rome. Sierpinski in four fold and six fold rosettes

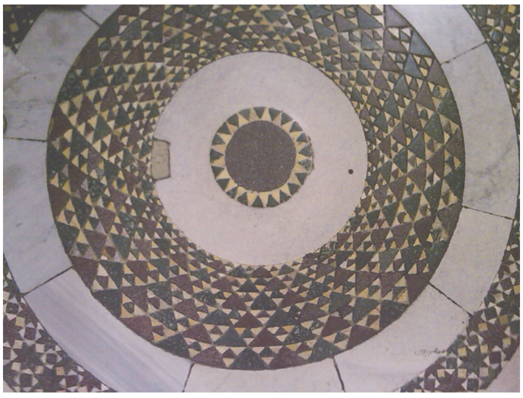
[1] Padre Crispino Valenziano, professor of Lithurgical Anthropology Pontificio Istituto Sant'Anselmo; Rome.
[2] The six-fold rosette is known as "seed of life" and present in various cultures; in its composition as lattice it was also studied by Leonardo da Vinci.

Fig. 8 SS. Giovanni e Paolo (13th century), Rome. Sierpinski in four and six fold circular wedges.

Fig. 9 Santa Maria in Cosmedin, Rome. Square lattice of rosettes

4.2 Later Cosmatesque works in Rome

Many visitors to Rome can notice carpets of the kind we illustrated, in famous places, such as the Sistine Chapel in the Vatican. The floors of the Sistine Chapel and of the Stanza della Segnatura (best known as Stanze di Raffaello, from the famous frescos) have floors dating to the 15th century. It is interesting to notice that while the carpets are rigorous, in this case we have seemingly attempts to draw isolated Sierpinski triangles, none of which is quite Sierpinski (fig. 10).



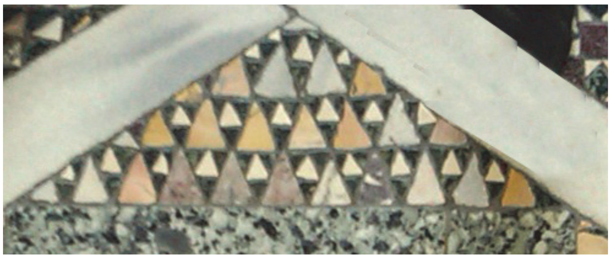


Fig. 10 Sistine Chapel, floor detail (left) Stanza della segnatura, Vatican (15th century)

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