

PERSISTENCE OF FORM IN ART AND ARCHITECTURE: CATENARIES, HELICOIDS AND SINUSOIDS

by Elisa CONVERSANO (I), Mauro FRANCAVIGLIA (I),
Marcella Giulia LORENZI (I) & Laura TEDESCHINI LALLI (I)

Abstract. The human mind tends to recognize numbers, shapes and forms in the external world. Geometric shapes persist in Art and Architecture from Prehistory to Modern Age. In this paper we report about an ongoing investigation into this persistence, starting from sinusoids and oscillations, catenaries and helicoids, chosen as possible organization centers of the many recognizable forms. The aim is to understand how, when and why this persistence of forms has accompanied the parallel evolution of Art and Science. Examples are chosen from Mesopotamian Art, Gothic, Islamic Art, Baroque and Modern Architecture.

Key words. Catenaries, sinusoids, helicoids

Mathematics Subject Classification: 01A45, 01A40, 01A35

1. Introduction

The human mind tends to recognize numbers, shapes and forms in the external world. Visual perception is transformed into proportions and geometrical shapes that artists and architects have used in all ages of History to produce their artworks, from paintings in caverns to modern visual artifacts, from carved stones to urban design, from elementary drawings to mechanically planned drawing and computer generated images. Forms have been recognized in Nature, used for their aesthetic value or because of their functionality, probably first understood at an emotional level, to be later elaborated at more conscious levels, gradually passing from emotion to formal theorization, and thereon to planning, by exploiting the predictive power of the formalized models. At times forms, by becoming reproducible and recognizable, lent themselves to visual symbolic communication, therefore linking to other realms and semantic purposes. In the framework of a more general project on the persistence of form (to which our papers [1],[2],[3],[4] refer) we present here our preliminary investigation about some specific curves and curved surfaces.

There is a potentially infinite family of «geometrical shapes» and structures that have crossed the ages and the cultures, from Prehistory to our days, from Orient to Occident, giving rise to what we can call the «persistence of forms» (see [5], [6]). The three families of curves and surfaces

investigated in this paper are catenaries and catenoids, sinusoids and sinusoidal concoids, as well as helicoids. All three curves: helix, sinusoid and catenary are non-algebraic, all three are subject to creating surfaces in more than one way, either as ruled surfaces or by revolution, as surfaces optimizing a potential, which are called “minimal surfaces”. The fact that these curves are non-algebraic in Cartesian coordinates is visually clear from their curved shape; Euclidean Geometry gave us a context to discuss straight lines, planes and essentially only conics. A thousand years later, the introduction of Cartesian coordinates allowed the study of curves as loci of solutions of specific equations. It was with the advent of infinitesimal calculus that we were finally able to discuss how curved a curve is, and, by the very same tools, to discuss forces along it [7].

2. Catenaries and Related Surfaces

In Geometry as well as in Physics a “catenary” is a planar curve representing the shape that an idealized hanging chain (or a heavy rope) assumes when it is supported at its fixed ends and acted upon by gravitational Galilean forces (i.e., weight). The name is derived from the Latin word *catena* (i.e., "chain"). The curve has a U-like shape and it is analytically related with the graph of the hyperbolic cosine *cosh*; it is superficially similar to a parabola, especially in small portions. The Cartesian equation of a catenary can be written as

$$y = a \cosh (x/a) = 1/2[\exp(x/a) + \exp(-x/a)]$$

where a is a real parameter, that can be interpreted as the ratio between the horizontal component of the tension on the chain (assumed to be homogeneous, so that the tension is constant) and the weight of the chain per length unit.

The history of the catenary is interesting in itself, as it provides an excellent example of the interaction between experimentation and theory. Guidobaldo Del Monte (1545-1607) was the first to define it, in his experiments within the flourishing mathematical school in Urbino (Italy) see [8]). The young Galileo was one of his students and protégées and went on with the study. They both incurred in errors, of slightly different nature: Guidobaldo erroneously thought that a hanging chain would assume the same curve, inverted around a horizontal plane, as the trajectory of a heavy body launched upward at an inclined angle with the vertical; he states that this curve is symmetrical, explains the physical reasons for it, and indicates that it resembles a parabola or a hyperbola, but that warns that it is better seen as a hanging chain. Galileo, as is well known (see [9]) correctly identified the trajectory of launched body as a parabola, but identified also the hanging chain as such, assuming the identity proposed by Guidobaldo, foregoing the *caveats* of Guidobaldo as to the differences between a hanging chain and a parabola (see [8]). He therefore assumed that a hanging chain has the form of a parabola; this was later disproved by Joachim Jungius (1587–1657) and published posthumously in 1669. Today we know that the initial mistake was in not noticing that the successive positions of a ball thrown in the air are not influenced by the constraint of continuity in the body, which is instead respected both for hanging chains and optimal arches. The correct equation was derived in 1691 by Gottfried Wilhelm von Leibniz, Christiaan Huygens and Johann Bernoulli. Huygens first used the term *catenaria* in a letter to Leibniz in 1690. The catenary was early used in the construction of arches (already at the time of pre-Greek and pre-Roman Architecture). We can now see this as an inversion of the hanging chain around a horizontal plane; in antiquity the curvature of the inverted catenary was in fact materially discovered and understood to be useful in the construction of stable arches and vaults (see [10]). Examples are found in Taqi-

Kisra in Ctesiphon (Mesopotamia – Fig. 1) while Greek and Roman cultures reverted to circular arches and semi-spherical vaults, where the curvature of a circle is much less efficient statically, but could well have a great drawing effectiveness. It somehow remained in Islamic Architecture but remained thereon forgotten in Europe for long time. It is supposed that its modern rediscovery was due to Robert Hooke - famous for his studies on Elasticity - who discovered it in the process of the rebuilding of St Paul's Cathedral. In 1671 Hooke announced to the Royal Society that he had solved the problem of the optimal shape of an arch, and in 1675 published an encrypted solution as a Latin anagram in an appendix to his Description of Helioscopes, where he wrote that he had found "a true mathematical and mechanical form of all manner of Arches for Building" [11]. This idea of an inversion with respect to the horizontal plane, as we saw, was already in the initial writings by Guidobaldo [8].



Fig. 1 An (almost) catenary arch in Ctesiphon, 6th Century BC

The catenary is very important in modern Architecture, as the ideal curve for an arch that supports only its own weight. In a good first approximation, when the centerline of an arch follows the curve of an inverted catenary, the arch is known to endure only pure compression, so that no significant torsional moments occur inside the material. When individual pieces form the arch and their contacting surfaces are perpendicular to the curve of the arch, moreover, it is known that no shear forces are present at the contact. No specific buttress is required, since the forces acting on the arch at the two endpoints are tangent to its centerline.

Antoni Gaudí (1852-1926) left an impressive mark that reveals his continuous interest in the role that Mathematics (and more generally the observation of Nature) plays in Art in general and in Architecture in particular. For the design of "La Sagrada Família" (see [1]) Gaudí studied and developed a new method of structural calculation based on models involving ropes and small sacks of lead shot (Fig. 2). The plan of the church was traced on wood and placed on a ceiling, with ropes hanging from the points where columns had to be placed. Sacks of pellets were hung from each arch formed by the ropes. These were in fact catenaric arches, as the Calculus of Variation dictates. He would take photographs of the resulting plastic model, shot from various angles and then turned them upside-down, so that the lines of tension formed by the ropes and weights would now indicate the pressure lines of the structure envisaged. In this way Gaudí obtained many "natural" forms in his work. Antoni Gaudí made extensive use of catenary shapes not only in the Sagrada Família but in most of his architectural work, as in the crypt of the Church of Colònia Güell (see [1]).



Fig.2 Study model made out of actual chains (left) and Model for the Construction, from the Museum of the Church (right, photo Lorenzi)

The surface of revolution of a catenary, called “catenoid”, is a minimal surface and is therefore the shape assumed by a soap film bounded by two parallel circles (as it was first proved by Euler in 1744).

In modern architecture catenaries have been exploited also in the employment of concrete, which allows an unprecedented levity [13].

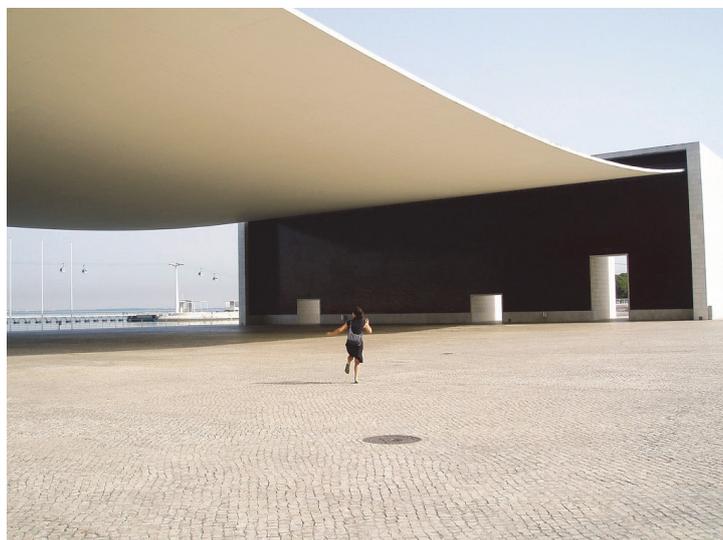


Fig. 3 Alvaro Siza Expo 1998, Lisbon, Portugal Pavillion

Alvaro Siza (born 1933), Portuguese architect, winner of Pritzker Prize in 1992 (the “Nobel of Architecture”), designed Portugal Pavilion for Expo 1998 (Fig. 3). In this building the idea of levity is realized by hanging chains and then filling by a thin layer of white concrete. Thus the entire area of 65x58 meters is topped by a geometrical shape molded in only one piece 20 cm thick.

The sections in two orthogonal directions are parallel catenaries and parallel lines, respectively; Gaussian curvature is therefore zero.

3. Sinusoids and Wavy Forms

Oscillating forms are ubiquitous, and we start our historical search, for the moment, at the Roman Imperial mosaic floors of the 2nd century AD; starting from the 1500's, we will then see that oscillating forms acquire more and more rigor and tri-dimensionality.

Roman imperial floors in mosaic share a waving motive used as framework, that allows following various shapes, depending on the needs (Fig. 4). Historians of Art call this motive “*can corrente*” (i.e., “running dog”) – or also “continuous wave” - and archaeologists¹ call it “braided polychrome bands”, thus pointing to different treatments in different scholarly environments. This motive is interesting for two apparent reasons: the connection with water, at symbolic and representation level, and the appearance of a braid, i.e. of a speculation about three dimensional features and relationships, as “above/under”, represented in a completely two dimensional artifact, and also worked in more intricate patterns, both in the imperial floors, and later in Celtic designs.



Fig. 4 Augusta Raurica, Switzerland, 2nd century AD

The waving motive resists throughout history in heraldic representation, where it acquires the name of “wavy” (or “*ondato*”); it is striking that it is maintained throughout various materials and centuries, as heraldic scopes dictate.

A waving braided motive is found again in the floors laid by the *Marmorari Romani*, 11th Century, in central Italy, imprecisely known as “*Cosmati*”, from the name of one of the most important families of *Marmorari*, who were the artisans who gained exclusive permission to reuse imperial marbles from the Pope.

¹ We acknowledge useful conversations with Daniele Manacorda.

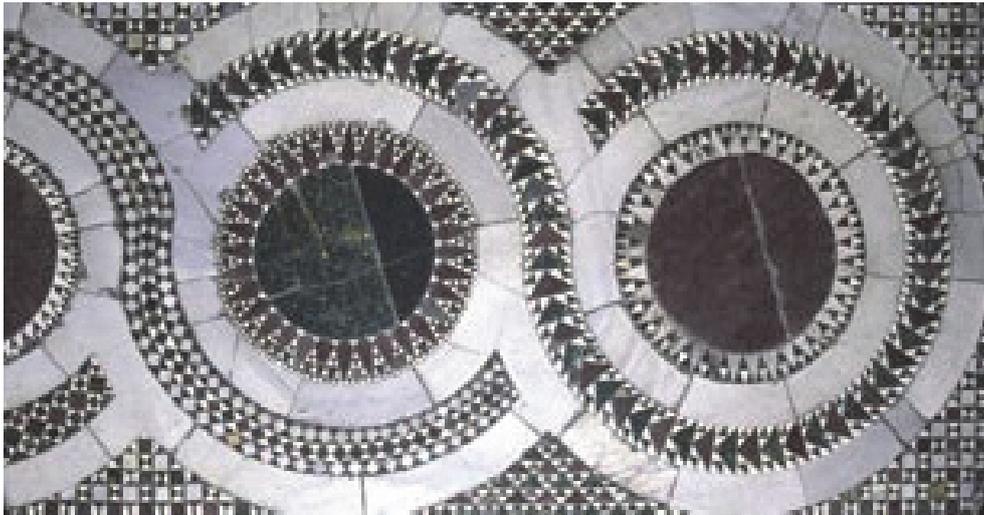


Fig. 5 Basilica of San Clemente, Rome

In churches with floors planned by the *Marmorari*, no water apparent symbol seems suggested; the motive, called “*guilloche*” in this case, is instead of high perceptive relevance, and has its role in liturgical itineraries (see Fig. 5). This can be better explained in modern mathematical terms: the believers would stand in areas covered by rectangles filled in multicolor marbles, all characterized by a finite group. The deacons celebrating would advance along the *guilloche* in the middle of the church, stopping where it formed “*quinconci*”² or where it crossed another *guilloche* [14]. A *can corrente* is characterized by an infinite group generated by one translation, and perceptively results in the imagination of movement along the direction of this translation.

One of the realms where wavy motives are found is certain gardens, and successively in modern landscape architecture. The first example that comes to mind is the black and white motive on the promenade of Copacabana beach in Rio de Janeiro, 1961. This pavement is part of the project of Roberto Burle Marx (1909-1994), a foremost landscape architect. The black and white stone motive is laid using traditional material and techniques of the Portuguese *calçadas*. *Calçadas* are made of square black or white stones, of side between 4 and 5 cm; the stones are therefore rather large; they rather resemble, in shape, that of the basalt cubes (called “*sampietrini*”) largely used in Rome. *Calçadas* are typically used for pavements of large public places, resistant to heavy use and easily replaced. In Portugal one can find many different motives, all very interesting. One of these motives is made of sinusoidal lines, already present in front of the Copacabana Palace, in the 1920’s, and Burle Marx uses it on the entire sea front, laid on a much larger scale, and making it an unforgettable signature of the landscape (Fig. 6).

Sinusoids and waving forms are often connected with the presence of water, as in the previous case of a waterfront, but also explicitly in Roman imperial mosaics. They make their appearance in landscape in the *giardino all’italiana* (“Italian Garden”) of the 16th Century, mostly as “water chains” (*catene d’acqua* - Fig. 7), bordering water flows.

² Padre Crispino Valenziano, Pontificio Istituto Liturgico Sant’Anselmo, Roma. Padre Valenziano, authority on Liturgy in a historical and anthropological perspective, relied this fact on special guided visits about Cosmati floors.



Fig. 6 Avenida Atlantica, Rio de Janeiro: 1930's (left) today (right), promenade by Burle Marx, detail



Fig. 7 Catena d'acqua o Cordonata del Gambero, Villa Lante, Viterbo

Sinusoid is still used, in quite different material, in open spaces, suggesting a garden, as in the work of Sandro Anselmi (born 1934), the leader of the Roman group of architects advocating the planning of “fluid spaces” in Architecture.

Dating from the 18th Century the sinusoid was so well understood that William Hogarth took it as a prototypical form of balance and elegance in his influential book “The Analysis of Beauty” [15].

When we look at surfaces, sinusoids can be again employed in architecture allowing levity. As another example of Gaudi's creative geometric geniality, we mention that in the “Schools”, again in Barcelona– designed in 1909 – he made use of small straight segments to construct curved surfaces. The roof is in fact a conoid with a sinusoidal section, robustly sustained by relatively thin walls (just 9 cm for a building of 10x20 meters and an eight of 5 mt...!) and also able to produce a good acoustic; see [16]. The form is thus a ruled surface, and the roof is composed of air-bricks (also known as *foratino*) laid in rows, long side along the straight lines ruling the conoid.

As to the historical development of the mathematical function, obviously sine and cosine were well known in ancient times when they were defined and used for astronomical measures, models and predictions. Tables of values were explicitly provided, and we today think of tables of values as “functions”. But if we are thinking of the form of the sinusoid, we are thinking about its graphical representation in Cartesian coordinates, which came much later. Moreover, nowadays sinusoids are widely used in signal processing and time series, a topic intimately linked to the realms where they

appeared as functions, i.e. in dynamics, where the independent variable is time, in itself a modern idea. We can date the study of oscillations at the beginning of the 18th century, with the pioneering works of a young Euler, later rephrased and reworked by Euler in the form that influenced today's treatment [17]. The study of oscillations is strictly linked with the study of vibrations, and Clifford Truesdell emphasized that the idea of dynamical equations was slow to emerge [18]. The oscillating aspect of a sinusoid prompted the problem of isochronicity, studied together with the simultaneous zero crossings of a solution.

4. Helicoids



Fig. 8 Santa Caterina, colonne tortili, Palermo (1566-1596)

Sinusoids can be used to design and represent the spiral columns or *colonne tortili*, a type of column much used throughout history and places (Fig. 8). It is rather straightforward, visually, to go from sinusoid to helicoids, via the *colonne tortili*, which have a sinusoid as profile.

A helicoid is a ruled surface obtained from a helix. It also appears naturally as a minimal surface when the bordering line is a helical. The helix has a very simple expression in polar coordinates, illuminating the fact that it was also quite easy to draw, by much the same reason, i.e. following angles as independent variables, for which mechanical drawing machines can be planned and built.

Generally the term “spiral stairs” is used to define a type of staircase. From a mathematical viewpoint, a spiral is a plane curve monotone in the angle ϕ if expressed in polar coordinates (r, ϕ) . Thus “spiral stairs” would not change in elevation, and would move toward a centre. The correct mathematical term for motion remaining at a fixed distance from a straight line, while moving in an upward circular motion about it, is “helix”.

In 16th Century helical stairs were much in use (Belvedere del Vaticano by Bramante, Villa Farnese and Palazzo Boncompagni by Vignola, etc...), and we chose some examples pointing to the three-dimensional static stability of the underlying internal curve (Fig.s 9, 10 and 11).

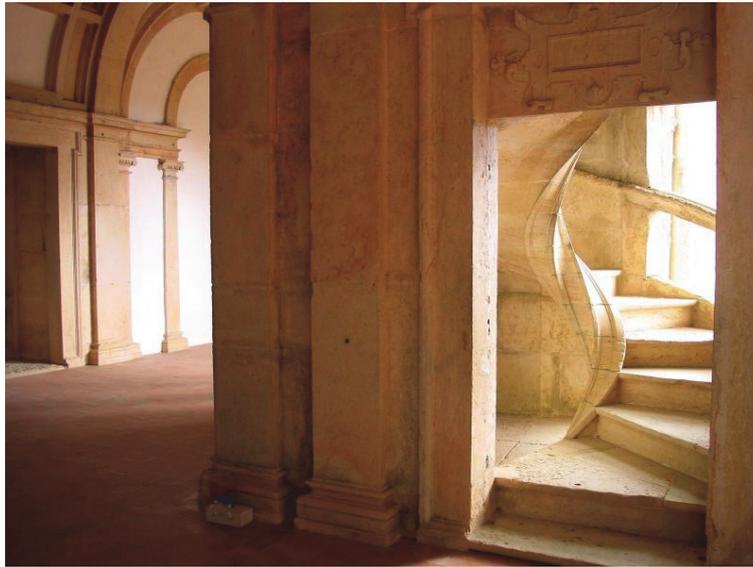


Fig. 9 Helical stair Tomar, Portugal (1160-1400)

Helical stairs rise around a pole without a column in the centre: the central support is made by a helix and this leads to a perceptual effect of emptying. While it is true that the stairs do not have a conventional straight central support, the tightly wound inner stringer (structure that supports the risers) functions as one. This twisted central support is built in fact like a giant spring. If the central helix is close enough, we can see something like a *colonna tortile* in the middle, instead of the cavity: that is the real support, the structure.

The gradual upward rotation is very elegant and was used also in narrow spaces, and as we saw sometimes even in big columns or in pilasters.



Fig. 10 F. Borromini, Helical stair in San Carlino, Rome, Italy 1640

We emphasize that all four cases share having no support in the middle cavity. Such a structure is so stable that the Louvre one is not anchored at walls either. In this special case, in which the stair is quite large, the steps are thickened toward the center to ensure stability fulfilling the function of balancing stresses.



Fig. 11 I.M. Pei Helical stair, Louvre, 1990 (left) and Gaudì stair (1920), Barcelona (right, photo Lorenzi)

Thus the helical stairs can be roundly released from walls and other supports like central columns or pilaster, returning in this way an image of lightness. Lightness and a rigorous sense of statics and usability were the inspiring values of the Italian school of architecture called Rationalism; we present here two examples of helicoal stairs, in fig. 12

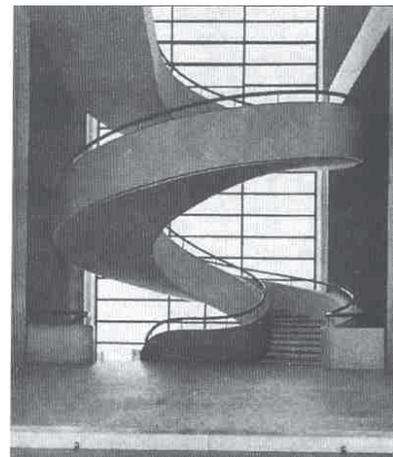
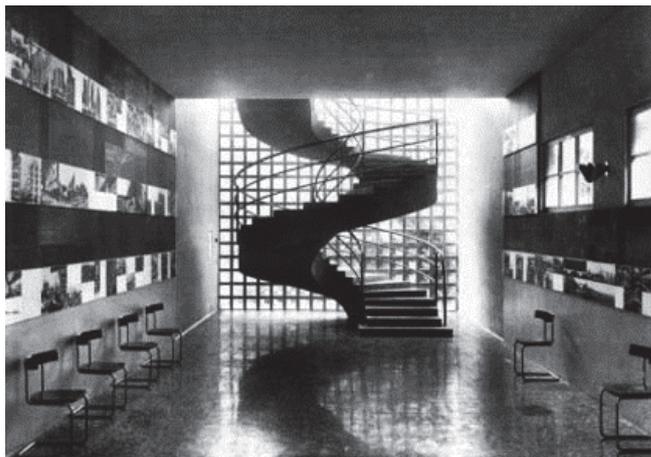


Fig. 12 Pagano - Buzzi Helical stair, VI triennale Milano, 1936 (left) and L. Moretti stair, Gil, Roma (1936), (right)

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Current address

Elisa Conversano, PhD student

Dipartimento di Studi Storico-artistici, archeologici e sulla conservazione,
Università Roma Tre,
p.zza della Repubblica 10, 00185 Rome, Italy
e-mail: elisa.conversano@gmail.com

Mauro FRANCAVIGLIA, professor

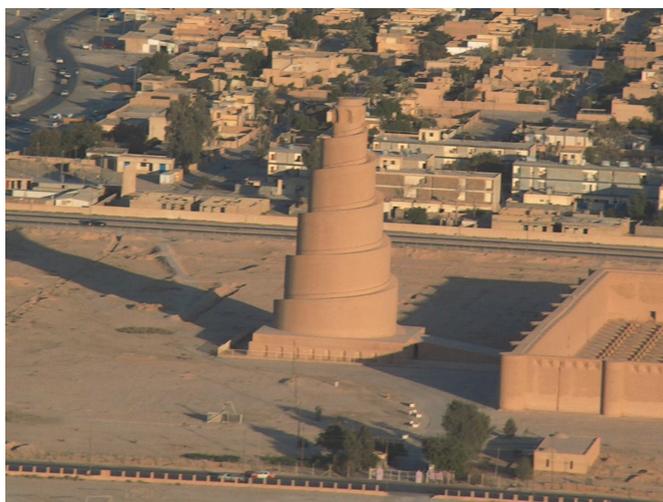
Laboratorio per la Comunicazione Scientifica,
University of Calabria,
Ponte Bucci, Cubo 30b, 87036 Arcavacata di Rende CS, Italy
and
Dep.t of Mathematics,
University of Torino,
Via C. Alberto 10, 10123 Torino, Italy ,
e-mail: mauro.francaviglia@unito.it

Marcella Giulia LORENZI, PhD

Laboratorio per la Comunicazione Scientifica,
University of Calabria,
Ponte Bucci, Cubo 30b, 87036 Arcavacata di Rende CS, Italy,
e-mail: marcella.lorenzi@unical.it

Laura Tedeschini Lalli, professor

Dipartimento di Matematica,
and
Facoltà di Architettura
Università Roma Tre
L.go San Leonardo Murialdo 1 I-00146 Rome Italy
tedeschi@mat.uniroma3.it



Minaret, Samara, Iraq 1st Century AD