



## **FORM AND ART OF CLOSED CREASE ORIGAMI**

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**Abstract.** We show some origami models appeared for the first time as student projects in Bauhaus around 1930. The same models are studied today by mathematicians and structural engineers for its geometric and mechanical properties and not yet completely understood. A brief discussion on origami axioms will introduce origami geometry.

**Key words:** Origami, hyperbolic paraboloid, folding, compass and straight edge.

*Mathematics Subject Classification:* Primary 51M15, 32Q05.

### **1 Introduction**

In recent years paper folding is becoming more and more a scientific tool and a field of research. We read about intersections between origami and geometry, technology, engineering. In this paper we will first introduce origami geometry and its axioms. Then we will focus on some models obtained by crease patterns (the set of lines on a sheet of paper which must be folded to obtain the final object) of concentric lines, squares or circles. This kind of pattern allows to create very beautiful shapes: indeed some of them are exposed in art museums. Moreover these shapes have a nontrivial geometry, which is not yet completely understood.

### **2 Origami axioms: the rules of the game**

In 1989 during the “First International Meeting of Origami Science and Technology”, (Ferrara, Italy) the mathematicians Humiaky Huzita and Benedetto Scimemi stated six axioms for origami geometry. A seventh axiom was added a few years later by Koshiro Hatori.

In the last 40 years much attention has been devoted to paper folding as a powerful scientific tool. The first person who focused that paper folding could be a powerful geometric tool was the Italian mathematician Margharita Piazzolla Beloch (University of Ferrara) around 1930 [6]. Benedetto Scimemi during the 80’s casually learned that there were some scientific papers by Margharita Piazzolla Beloch where she showed some of the rigorous algebraic aspects of paper folding. She explained also how some of the *impossible geometric problems of antiquity* are indeed possible using paper folding. Actually she realized that paperfolding can be used to solve equations of third degree, so also proved that origami geometric proofs are more powerful than compass and straightedge ones. In 2011 Tom Hull, one of the most acknowledged mathematicians who work on

origami at the moment, wrote a note on American mathematical monthly presenting her original proof of 1936 [6]. Nevertheless the name of Margharita Piazzola Beloch is rarely quoted when speaking about origami axioms and geometry.

Ancient greeks developed many geometric constructions, but it is well known that they were not able to solve some of them, known as *impossible geometric problems of antiquity*. In particular trisection of an angle, cube duplication<sup>1</sup>, circle squaring<sup>2</sup> are examples of such problems. After many years of attempts to prove them, mathematicians changed point of view: they tried to prove that these construction were impossible to do with ruler and compass. These proofs of impossibility were given using algebraic methods, two thousands years later.

Also using algebraic methods it was proved that quadratic equations can be solved with compass and straightedge [2]. Origami geometry is in some sense more powerful since it can solve equations of degree three; furthermore the class of polygons constructable by paper folding is wider than the one of polygons that can be constructed with compass and straightedge.

One of the main reasons of this difference relies in the fact that the “rule of the game” in compass and straightedge construction says that the ruler is not *graded*: it can be used only to connect points with lines. On the other hand, the edge of a sheet of paper can be *graded*! You can make a “pinch” (a tiny fold, used just to mark a point) in the center to mark the middle point. If we allow a ruler to have two marks, angle trisection could be achieved. There is a construction, due to Archimede [2], of angle trisection done with compass and a ruler with two marks.

The axioms of origami establish the rules of the game, and the game consists in doing geometric constructions by folding paper. Origami considers both straight and curved folds, but when dealing with geometric constructions creases must all be straight. Straight folds are lines, points are intersection of at least two creases (vertexes on the crease pattern).

The edge of the sheet of paper works as a ruler, and so do all the straight creases. Here is a list of the seven axioms [7]

1. Given two points  $P_1$  and  $P_2$  we can fold a line connecting them.
2. Given two points  $P_1$  and  $P_2$  we can fold  $P_1$  onto  $P_2$ .
3. Given two lines  $l_1$  and  $l_2$  we can fold line  $l_1$  onto  $l_2$ .
4. Given a point  $P_1$  and a line  $l_1$  we can make a fold perpendicular to  $l_1$  passing through the point  $P_1$ .
5. Given two points  $P_1$  and  $P_2$  and a line  $l_1$  we can make a fold that places  $P_1$  onto  $l_1$  and passes through the point  $P_2$ .
6. Given two points  $P_1$  and  $P_2$  and two lines  $l_1$  and  $l_2$  we can make a fold that places  $P_1$  onto line  $l_1$  and places  $P_2$  onto line  $l_2$
7. Given one point  $P$  and two lines  $l_1$  and  $l_2$ , there is a fold that places  $P$  onto  $l_1$  and is perpendicular to  $l_2$

For example, tracing a line with a ruler corresponds to make a fold. The first axiom tells that there is always a line (fold) that connects two distinct points; the second that folds can do symmetries: in fact given two points  $P_1$  and  $P_2$ , the fold which brings  $P_1$  on  $P_2$  realizes the symmetry axis of the segment joining them.

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<sup>1</sup> given a cube, try to construct another cube which has volume two times the volume of the given one

<sup>2</sup> constructing a square which area is equal to that of a given circle

### 3 Closed curved creases

A square of paper can be folded to a shape having negative curvature, as a hyperbolic paraboloid. The folds are closed concentric curves, and the creases alternate mountain to valley. In origami terminology, a valley fold is when the paper forms a “V”. If you turn the fold upside down, you get a mountain fold.



Fig. 1 - 2. Hyperbolic paraboloid folded by Paola Magrone (left), and by the students of the School of Architecture (right), Università degli Studi Roma Tre, during the workshop OrigAMI, 7-8 October 2014

The crease pattern of this origami is very simple and so is the folding process. Indeed this origami is very much studied because its geometry is not yet completely understood.

The first pictures, as far as the author knows, of this paper sculpture are dated 1928 and are related to students projects in Bauhaus [4]. Bauhaus was an art school active in Germany from 1919 to 1933 which had a very significant influence on architecture, art and design of XX century. Joseph Albers, artist and professor in Bauhaus, used to give preliminary courses where students made experiments with many different materials: wood, glass, fabric, and of course paper. They were encouraged to investigate the peculiar features of every material; then learn how to use each one in the best way in design.

This is how Albers addressed to his students before starting an experimentation on paper:

*'Ladies and gentlemen, we are poor, not rich. We can't afford to waste materials or time. [...] therefore we have first to investigate what our material can do. So, at the beginning we will experiment without aiming at making a product. At the moment we prefer cleverness to beauty. .... I want you now to take the newspapers ... and try to make something out of them that is more than you have now. I want you to respect the material and use it in a way that makes sense – preserve its inherent characteristics. If you can do without tools like knives and scissors, and without glue, [all] the better.'*<sup>3</sup>

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<sup>3</sup> From “Josef Albers, Eva Hesse, and the imperative of teaching” by Jeffrey Saletnik, In Tate’s online research journal

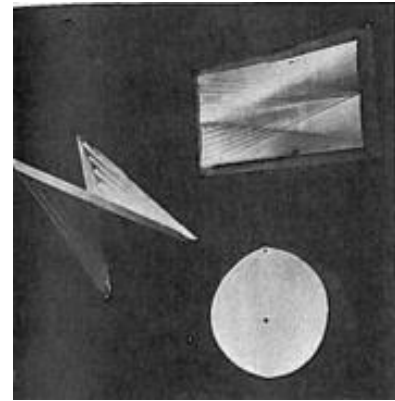
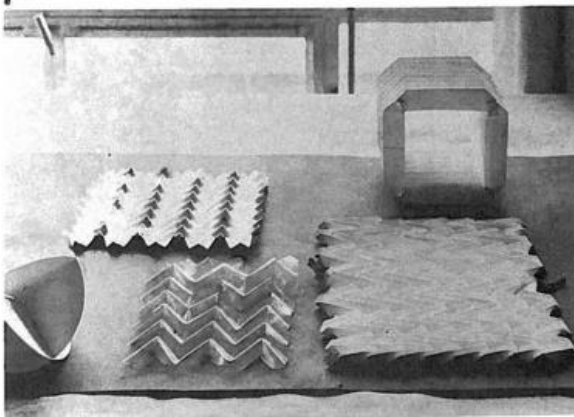


Fig. 3 - 4. Black mountain paper studies, 1937-1938, course held by Albers , in Esther Dora Adler's thesis "*A New Unity!*" *the art and Pedagogy of Josef Albers* [University of Maryland, 2004].



Fig. 5. Teaching of the preliminary course with Josef Albers and students, Black Mountain College, 1944. Photograph by Josef Breitenbach

The geometry and properties of the crease pattern of the paraboloid is described in [3]. The authors proved that the paraboloid “cannot be folded with the usual crease pattern”. This means that when folding some creases are added, not voluntarily, so in the final object there are more creases than in the original pattern. In the article an alternative crease pattern is showed. Following the same kind of pattern (concentric curves, alternating valley and mountain fold) one encounters very interesting and beautiful objects. For example starting by a circular sheet of paper with hole, like an annulus, and folding along one closed curve, you obtain this shape:



Fig. 6. Curved fold origami structure with three prescribed curves, in [5]

Paper finds in some way a configuration of equilibrium: the flat portions want to stay flat and the creased parts want to remain curved. So this concurrence of different forces leads paper to twist. The structural mechanical behaviour and the geometry is not yet completely understood. A first description of the mechanics is given in [5].

When folding this models, at the end of the process, one realizes that paper “wants” to twist and places itself in an equilibrium position, that, in the end, creates these magical shapes. An American mathematician, Erik Demaine, folds paper discs along closed lines, and creates paper sculptures. Some of them are in the permanent exhibitions of Museum of Modern Art in New York

The hyperbolic paraboloid of picture 1-2 is very much studied for many reasons. One of them is that it is an example of a non *flat* and non *rigid* origami. The most studied class of origami for industrial application are the so called *flat* and *rigid* origami. An origami is *flat* if it can be represented in a plane, in other words if we neglect the thickness of paper, the final object can be considered two-dimensional. An origami is *rigid* if it can be folded rigidly, in other words the flat parts between the creases are not bent during the folding process. A flat and rigid origami can be constructed replacing the flat parts with rigid panels and creases with hinges, and can be packed in less space since it is flat when folded.

However, the hyperbolic paraboloid is neither flat nor rigid. During the folding process one realizes that flat parts, that are very long trapezia, do bend. In [7] it is proved rigorously that this origami is neither flat nor rigid.

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