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Abstract. Reconstruction of the original aspect of ancient floors is a classical problem for archaeologists and restoration architects. Via the mathematical theory of periodic tessellations we reconstruct the original aspect of floors compatible with the fragments on site at theTrajan Markets, Rome. Our solution is unique under standard assumptions on regularity of the patterns. The experts had previously considered the fragments of insufficient information.

The result applies the twentieth century mathematical thought "symmetry=less information". This theorem has been largely used in visual analysis and classification; here is a first instance of its use for reconstruction, i.e. for its predictive power.

Key words: Group action, isometry, visual analysis

Mathematics Subject Classification:05B45, 58D19

1 Introduction

In this study we reconstruct completely the original aspect of the floors of the tabernae in the hemicycle of the Trajan Markets in Rome, Italy (fig.1). The reconstruction is based on the fragments remaining on site. Our reconstruction is unique, under assumption of regularity of the patterns, widely accepted by archaeologists; it is obtained via a theorem of the twentieth century about periodic tessellations of the plane. Remaining fragments are often small and separated and this yields problems of restoration and conservation of the floors. The algebraic-geometric approach succeeds in reconstructing the global aspect of the floors starting from much less local information than classically sought.

The study of reconstruction of floors, a classical problem for archaeologists and restoration architects, evidently stems from the hypothesis that missing parts can be filled-in starting from the information still available on place; i.e. there is an hypothesis of redundancy or recurrence of motives, techniques, materials, in either the very same floor, or in contemporary ones. Our method only deals with the geometrical repetitions of the specific floor under study. Our criterion for accepting or rejecting a reconstruction is based on the possibility to link, via a rigid motion, far away fragments. Under this objective criterion, we obtain only one motive compatible with each *taberna*.

Fig.1 The Hemicycle of the Trajan Markets

To our knowledge, there are no studies exploiting the mathematical theory of tessellations to reconstruct or fill-in for missing parts in artifacts, while it has been the base for several analytical studies of visual artifacts in their integral state of conservation [1]. We owe much to these seminal studies, leading to open problems in our own fields, as is the recent discovery of aperiodic tessellations in Islamic artifacts [2].

The method we propose is general, not linked to the historical period or material of the particular floors we investigate; in principle, it can be applied wherever rules of repetitions are apparent, as it is based on the classification of groups, i.e. the composition of such repetition rules. The general application of the method obviously depends on the extension of the missing parts.

2 History and architecture.

The name “Trajan Markets” denotes the buildings along the hill foot of Quirinale, behind the Trajan Forum: together with the Republican Forum, it defines the center of civic life in ancient Rome. The structure has been probably planned by Apollodoro, also author of the Trajan Forum (II sec. A.D.), for the enlarged needs of the Imperial Rome.

The *tabernae* we study lean directly against the rocks of the Quirinale, at the ground floor of the Great Hemicycle. These are eleven frescoed rooms, with barrel vault, travertine jamb, architraves and doorstep, and mosaic floors. Their dimensions are 3x2mt, with an average height of 5 mt. The mosaics are in *opus tessellatum*[1], with stone *tesserae* of 1cm. The motive lays in a central field joined to the walls by a rectangular frame. The field and the frame have varying proportions.

The motives employed are black and white geometric, according to the decorative traditions of Imperial Rome. The urban expansion determined an increasing demand of floors, to living ends and for public buildings (therms, basilicas, and commercial use) [3]. Black and white mosaic responded to the emerging needs with modular decorations easily extendable on different scales [4] .

3 The mathematical theory of plane tessellations, our method.

Our method of analysis and then reconstruction is based on the mathematical theory of plane tessellations, i.e. of plane configurations in polygons, without gaps or superpositions. The study of plane tessellations, stemming from algebraic geometry and algebra [5] has several applications. For a visual reference, we consider “plane tessellations” the decorative motives of floors and walls of many buildings, particularly characterizing the intricacies of Islamic Art. From the famous palace of Alhambra in Granada, Spain, to the decorations and friezes of the “quartiere Coppedè” in Rome, artifacts display different historical and material features, but a common compositional criterion, that can be analyzed as an instance of plane tessellation. Repetition of motives is common to all such examples. Repetition, in this case, mathematically translates to invariance under rigid motions on the plane, or isometries of the plane. How many groups of rigid motions can be defined on the plane? This question, arisen in crystallographic context in the mid 1800’s, got complete mathematical treatment in 1924, by Polya. Coxeter reports independent discoveries of the classification [6]. The answer is there are only 17 laws of periodic repetition in the plane. Whomever approaches classification of any type is greatly helped by such kind of theorem, guaranteing the very finiteness of the classification. Moreover, the proof of the theorem gives the criteria for the classification and comparison between different studies. Obviously, there are infinitely many possible variations that can be obtained by changing the basic motive. In this paper, we look at this “compositional” level of analysis.

Definition: An object is “symmetric” if it is invariant under an isometric transformation of the space.

[1] O*pus tessellatum* is the Roman floor mosaic, consisting of black and white square tesserae varying from 1 to 2 cm.

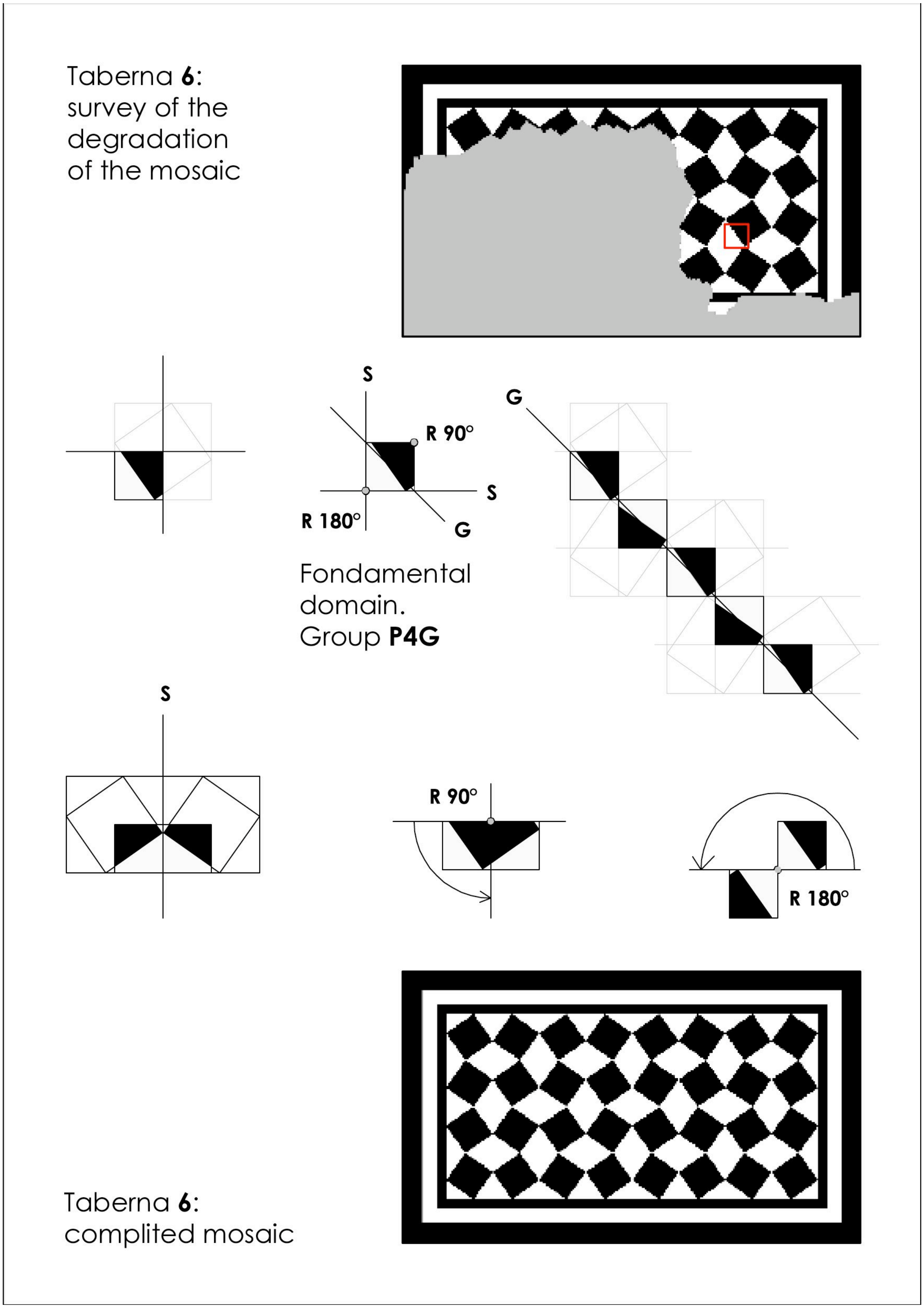

Fig.2 Analytical study of the fundamental domain and fill-in: case of glide reflection

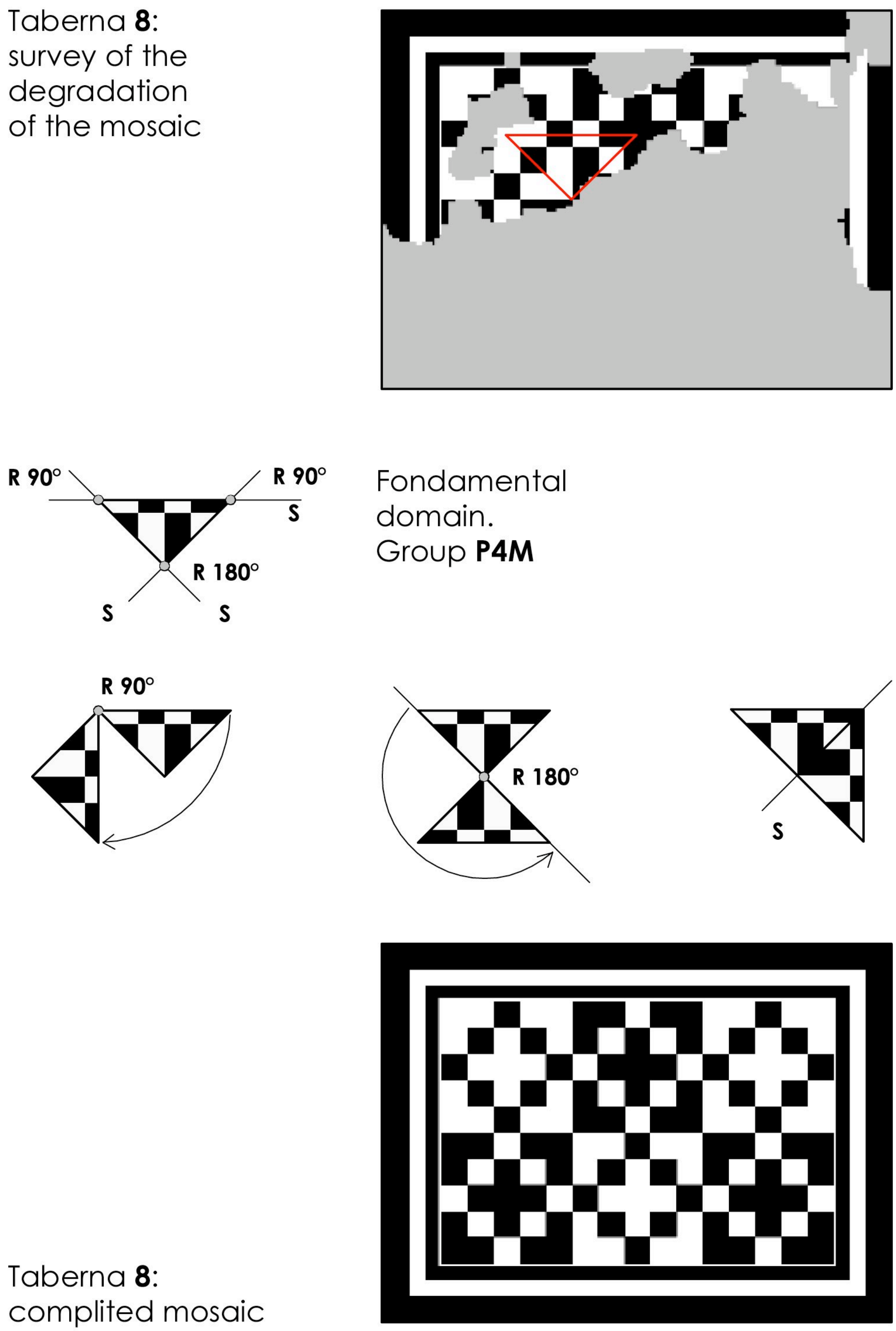

Fig.3 Analytical study of the fundamental domain and fill-in on a very fragmented floor

In our case the space is the 2-dimensional plane, and therefore possible isometries are rotations, translations, reflections and glide reflections. A planar motive is characterized by the set of isometries under which it is invariant, and they form a Group.

The important tool we used from the theory of tessellations is that of "Fundamental domain", or the minimal region that contains all graphic information needed.

Definition: a Fundamental Domain of a group of isometries is the quotient space of the plane with respect to the equivalence relations established by the Group of isometries.

A Fundamental Domain, visually, is a connected region, together with rules of behavior at its border. The region contains all graphic information, and the border rules its repetitions to fill the plane (fig.2,3). We worked out this analysis for all floors with fragments on place.

The advantages of looking for the Fundamental Domain in our case are:

- It is usually quite smaller than the region searched by archaeologists, because it retains more possible isometries, and therefore more information in a smaller space. It therefore has more chances to be in an integral state;.
- All these groups are finitely generated. We can search the Fundamental Domain for each generator of the group, over disconnected fragments, and recompose it over compatible fragments.

Definition: A Group of isometries of the plane defines a periodic tessellation if it splits into a finite (periodic) part, and an infinite part generated by two linearly independent translations.

Visual characteristic of a motive invariant under a periodic tessellation group is a lattice defined by the translations, and a rectangular area whose borders are the two translation vectors. This rectangular area is a representation of the fundamental domain of the subgroup of translations. Each time we analyze an actual floor, we imagine it to be indefinitely continuable on the plane. Taking into account the other isometries of the group, and quotienting, necessarily reduces the area.

We find in these floors 4 out of 17 possible groups. More precisely, these lattices only show angles of π or $\pi/2$, which is a strong visual information.

4 Survey sketches study.

Our survey study developed in phases, focusing on material details as it proceeded. We composed for each floor an analytical description containing: the situation of the taberna with respect to the hemicycle, a photo survey of the degradation, at various scales; the graphics comparing the degradation survey and the floor completed by applying the symmetry group. An example of this description is in fig.4,5. This enabled us to recognize different degradation situations, and arrange our search accordingly:

- Floors with relevant parts of integral mosaic, and internal holes of varying extension. In this case the aim is to fill the holes.

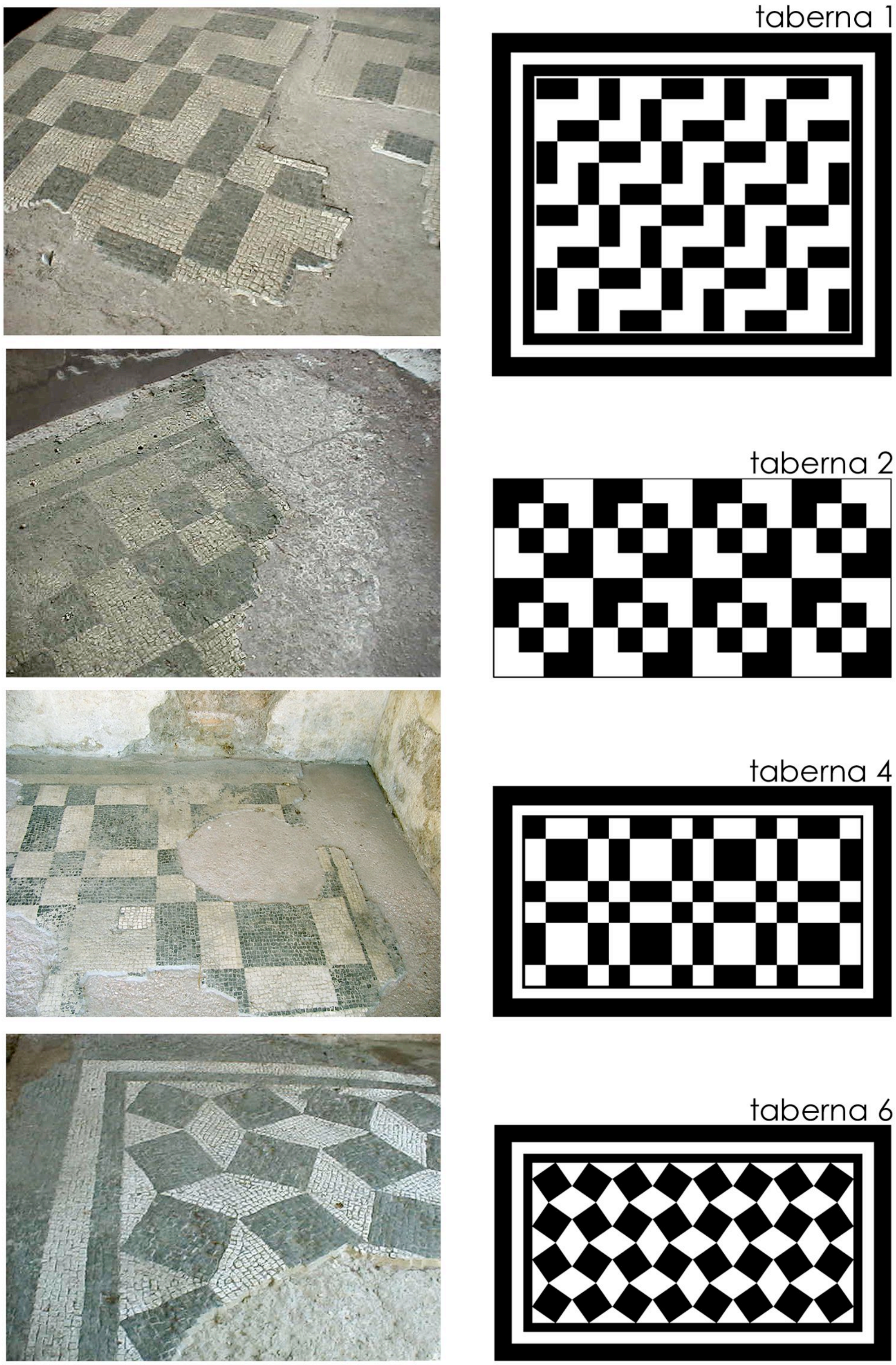

Fig. 4 Survey of degradation, and reconstruction of the *tabernae* 1, 2, 4, 6

Fig.5 Survey of degradation, and reconstruction of the *tabernae* 7, 8, 9, 11

- Floors whose remaining parts are situated in disconnected regions, at times quite fragmentary; in this case the action of the group on the graphic information contained in the fundamental domain bridges onto the distant fragments.
- Floors with very scarce remaining fragments. These seemed of impossible reconstruction. In fact, our method proved especially effective in this case, as the Fundamental Domain can be found in its disconnected form, in different areas of the floor, separately for each generator.

The video [7] display graphic animations for each floor, starting from the small region of the Fundamental Domain and reconstructing the motive under the action of the group.

5 Conclusions.

We present a method for reconstructing univocally a fragmentary floor, starting from objective information contained in the actual floor. This method is based on the theory of Plane Tessellations, and the search of the fundamental domain. As long as we know, this is the first instance of use of such theory for reconstruction purposes.

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References

[1] Peréz-Gòmez, R., *The Four Regular Mosaics Missing in the Alhambra* in Comp. and Math. with Appl. 14 (1987) 133-137

[2] Lu, P.J., Steinhardt, P.J., *Decagonal and Quasi-Cristalline Tilings in Medieval Islamic Architecture* in Science 23 Febbraio 2007: 1106-1110

[3] Fiorentini Roncuzzi, I., *Il mosaico. Materiali e tecniche dalle origini ad oggi*, Ravenna 1990, pag.22-58.
Farnet, M., *Glossario tecnico-storico del mosaico*, Ravenna 1993, pag.195-200.

[4] Vitruvio *De Architettura*, VII, I, 1-4, Plinio il vecchio *Naturalis Historiae* XXXVI, (186-187)

[5] Nikulin,V., Shafarevich, I. *Geometries and Groups* Springer V. 1987
Armstrong, M. *Groups and Symmetry* Sprinter 1988

[6] Fedorov Zapiski E.S., *Imperatorskogo S. Peterburgskogo Mineralogicheskogo Obshchestva* (2), 28 (1891).
Polya G., Niggli, P., *Zeitschrift fur Kristallographie und Mineralogie* 60 (1924), pp. 278-298

[7] Carlini A., Conversano E., Sabatini V., Tedeschini Lalli L., catalogo della mostra *Immaginare Roma antica*, a cura di M. Forte, L. Ungaro, 2005

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