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COMPOUNDS OF HELICAL CURVES: MEDIEVAL TWISTED COLUMNS

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Abstract. Twisted columns are ubiquitous in cult places, but classical treatises discuss only some specific geometric models. We address medieval columns, which elude this description, and introduce a parametric model mathematically describing a great variety of columns, and their differences. The model is validated through 3D survey. We describe the variety of columns of the Cloister of Saint Paul outside the walls,(A.D.1204-45), in Rome. When a column looks more wrought or torqued, it is obtained by compounding several different helices.

Keywords: cosmatesque art, elices, twisted or cochlear columns, Frenet vectors, point-cloud

Mathematics subject classification: Primary 00A67; Secondary 00A66, 01A35, 53A05, 62H35, 65D18.

1 Introduction and some history

Following in the study of *persistence of forms* initiated in [4], we report of an ongoing study into twisted columns (also known as cochlear, spiral, helical, tortile or solomonic for reasons we will see) . Twisted columns are ubiquitous in cult places, where their upward motion acquires symbolical meanings, together with their necessary intertwining. To us it seems that some columns are characterized by an imaginary of *rigid body*. Others, seem rather characterized by an imaginary thinking of an *elastic body*. These latter ones are all found historically in medieval columns, and not in later ones. The outward look of a twisted column can vary greatly, and there is no classification or description available, to our knowledge. Common to all twisted columns is an intention to spiralling, that contrasts with the need for support usually associated with columns. In fact, their support function should be discussed case by case, and seems in general not essential [3]. Historically, or possibly mythically, twisted columns are reported to have been at the entrance of the Temple of Salomon. Certainly there was a use of spiral columns in byzantine churches. Emperor Constantine is reported to have brought to Rome the two columns of the Solomonic Temple, thus initiating a long tradition. This tradition, though, is rather documented and studied from the Baroque era and afterwards. The byzantine columns could have well travelled west together with the Benedictine monks in whose medieval cloister today we find more complicated cases of twisted columns, not well

Fig. 1. A panoramic view (photo Falcolini) of some columns in Saint Paul's cloister.

studied, and that we think worth investigating. We make this conjecture based on the places we found the interesting compound columns.

Mathematically, the compositional core of a twisted column is an helix. Such an helix, though, can be used in different ways in the process of designing and building a column. In particular, as mathematicians, it is quite apparent to us that some columns are built and characterized by an imaginary of *rigid body*, so that it can be feasible to describe a way to sculpt them. Others, seem rather built and characterized by an imaginary thinking of an *elastic body*, so that it is in principle quite impossible to describe how to sculpt them in stone. These latter ones are all found historically in medieval columns, and not in later ones. The outward look of a twisted column can vary greatly, and there is no classification or description available, to our knowledge. We are interested in medieval columns, which are not discussed in later treatises, so that no geometric description is available.

As a document of this lack of geometrical description until the early 1500's, we cite from mathematician Luca Pacioli, who still in 1509 wrote:

Dove ora se trovino colonne piú debitamente fatte per Italia da li antichi e ancor moderni.

Cosí medesimamente se dici de quelle de Santo Pietro e Santo Paulo extra muros; ma quelle che sonno 'nanze a l'altare de Santo Pietro, fatte a vite, forono portate de Jerusalem, tratte del tempio de Salamone, de le quali l'una ha la immensa virtú contra li spiriti mali, comme *piú volte ho veduto, per lo suo santissimo tatto che feci el nostro salvatore Jesu Christo. De queste non si dá norma, se non quanto a loro altezza e basa e capitello, ma non de tal viticcio, peroché pó essere piú stretto e piú largo, a libito de l'ochio.*

Where in Italy can be found columns properly made by the ancients and by the modern.

And so in the same way, if you talk about those of St Peter's and St Paul outside the walls; but those that are in front of the altar of Saint Peter's which are done as vine, were brought from Jerusalem, taken from the temple of Salomon, one of which has immense virtue against the evil spirits, as I have several times seen, from the mostly saint touch that did our Savior Jesus Christ. Of these, it cannot be found a norm, other then about their height, basis and capital, but not about such a tendril, as it can be leaner and larger, ad libitum for the eye.

Twisted columns are described geometrically in treatises starting from that of Vignola (1562) [9], onward to that of Blondel (1777) [2]; in all of them twisted columns are described in vertical sections with ondulated lines vertically parallel. Such lines, and the verbal description, imply horizontal sections are circles of constant radius. The center of the circle moves along a cilindrical elix. A complete table of comparison among drawings in different such treatises, can be found in the exhaustive study by Tuzi [8].

This study stems from the observation that twisted columns built before such rationalization of the treatises, have a much more "torqued" look, not merely chiral. Therefore, we decided to investigate whether one can introduce variables to describe such complication.

We introduce a parametric model, in order to make explicit the movement around a cylindrical helix, taken as abstract reference. In medieval columns several helixes can be perceived by the onlooking eye, so that not only one helix is abstracted by the visual perception. In fact, by mathematical modelling, we find that medieval columns accommodate a greater number of possibilities to be composed around a principal abstract referential helix.

Mathematically, such modelizations are exploited in elegant and new compositions using the Frenet reference frame along a principal helix, in [6, 7]. In such studies the surfaces are the ones actually composing the hierarchies of helices, while we also consider the surface connecting two different helices.

One important feature of our model is to differentiate among columns by accounting for qualitatively different horizontal sections different models have. Moreover, the model we introduce distinguishes simple and compound columns. Compound columns are visibly made of ribbons. Our model describes each ribbon by vectors moving along the helix, remaining fixed in the Frenet frame. For an account on differential geometry of curves, see, for instance [1].

2 Our Mathematical Model for a Twisted Column

2.1 The abstract reference helix

A cylindrical helix is a space curve whose tangent vector is at a constant angle with the axis of the supporting cylinder, and lying on the surface of the cilinder. A parametric equation of an helix on a vertical cylinder of radius r can be written as:

$$\vec{P} = \vec{P}(\theta) = (r\cos\theta, r\sin\theta, p\ \theta) \tag{1}$$

so that the helix pitch is computed as p/r . The Frenet frame (tangent, normal and binormal unit vectors $\underline{t}$, $\underline{n}$ and $\underline{b}$) for this helix is given by:

$$\begin{aligned}\underline{t} =& \frac{1}{\sqrt{r^2+p^2}}(-r\sin\theta, r\cos\theta, p)\\ \underline{n} =& (-\cos\theta, -\sin\theta, 0)\\ \underline{b} =& \frac{1}{\sqrt{r^2+p^2}}(p\sin\theta, -p\cos\theta, r)\end{aligned} \tag{2}$$

and the curvature κ and torsion τ are constant along the curve

$$\begin{aligned}\kappa =& \frac{r}{r^2+p^2}\\ \tau =& \frac{p}{r^2+p^2}\end{aligned}$$

This helix is in the following taken as abstract reference for constructing different types of twisted columns.

2.2 The fixed radius column

From the appearance of historical treatises on (i.e. from 1562), twisted columns are designed by the motion of a circle of fixed radius R, whose center moves along the reference helix (1). The circle lies on a horizontal plane, i.e. a plane orthogonal to the helix's axis. Different treatises and different columns, in this type, only differ for the relative ratios: horizontally, of helix radius r to circle radius R and, vertically, of helix radius r to the helix's pitch $\frac{p}{r}$. An illustration of different possibilities is given in Fig. 2, using the model of the preliminary study [3]. Common to all treatises is the imagery of a horizontal disk sliding along the helix. This imagery is not sufficient for more twisted columns.

2.3 Compound twisted column

The fixed radius model cannot possibly account for all the columns present and recurrent in medieval cloisters in central Italy. Such columns, as mentioned, are found in Benedictine and Cistercensis cloisters, and probably their style comes from the Orient [8]. Already in the cloister of Saint Paul outside of the walls, we can see a variety, as documented in Fig. 1, that cannot be described in terms of a horizontal sliding disk, no matter what parameters. Here we introduce a more complete parametric model, accounting for columns of several types seen in this Figure.

The eye perceives "ribbons" when looking at these columns. The main new feature introduced in our paper is a geometrical description and treatment of these "ribbons", which constitute the external surface of the column.

A *vertical ribbon* on the cylinder is generated by the translation of the reference helix (1) in direction parallel to the cylinder axis: its parametric equation is therefore the linear combination

$$(1-t)(r\cos\theta, r\sin\theta, p\ \theta) + t(r\cos\theta, r\sin\theta, p\ \theta + h) \tag{3}$$

in the variable $t \in (0, 1)$ and θ (see Fig. 3).

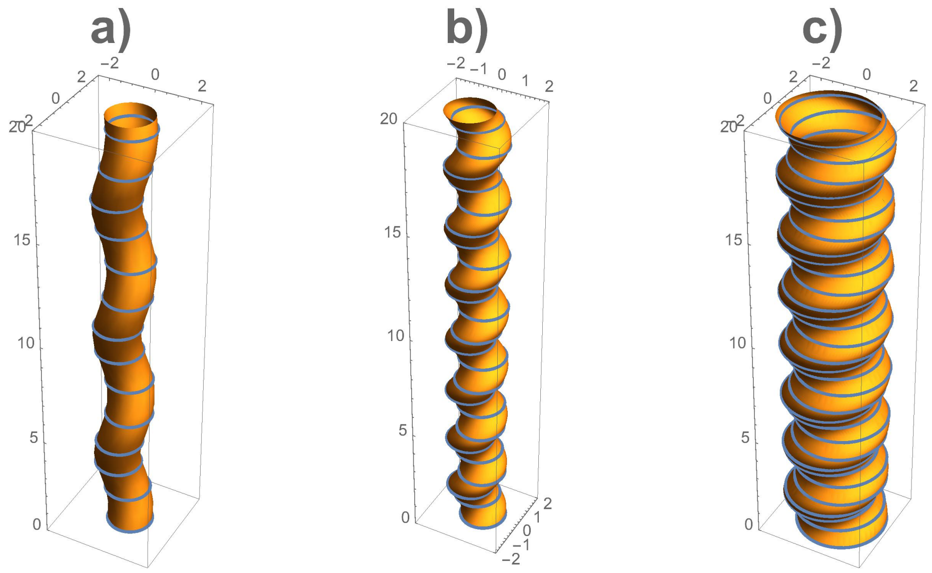

Fig. 2. Parametric model in Mathematica of three cases as described in the treatises post-1600: $r = 0.3$ in all cases; a)$R = 1$ and $p = 1$; b)$R = 1$ and $p = 4.2$, c)$R = 2$ and $p = 4.2$.

More general ribbons on the twisted column are generated by segments P_iP_j of a polygonal chain at a fixed orientation with respect to the Frenet frame of the reference helix (1): since most of the columns have four ribbons, a parametric model of the column can be given as a system of four equations of the kind

$$(1-t)\vec{P}_j + t\vec{P}_{j+1} \tag{4}$$

with

$$\vec{P}_j = \vec{P} + c_{1,j}\underline{t} + c_{2,j}\underline{n} + c_{3,j}\underline{b}$$

for constants $c_{1,j}, c_{2,j}, c_{3,j}$ and the Frenet versors of (2).

In order to validate this model, we surveyed the actual three dimensional columns. We used the software Photoscan, to obtain a point cloud of each column from a sequence of photos taken around the columns. We then studied models for sections of the point cloud: see Fig. 5 for the strange looking horizontal sections of our model.

First of all, we note that horizontal sections of the point cloud are not circles, as can be seen in Fig. 6.

Next we note that the plot of one of the horizontal sections of our model seems to better agree to the horizontal section of the point cloud than the circle (see Fig. 6).

In seeing ornamented columns, such as those shown in Fig. 4 the onlooker first perceives the white curves. Our model implies that these curves lie on the borders of ribbons and therefore are all helices. Each of the helices can lie on cylinders of different radius.

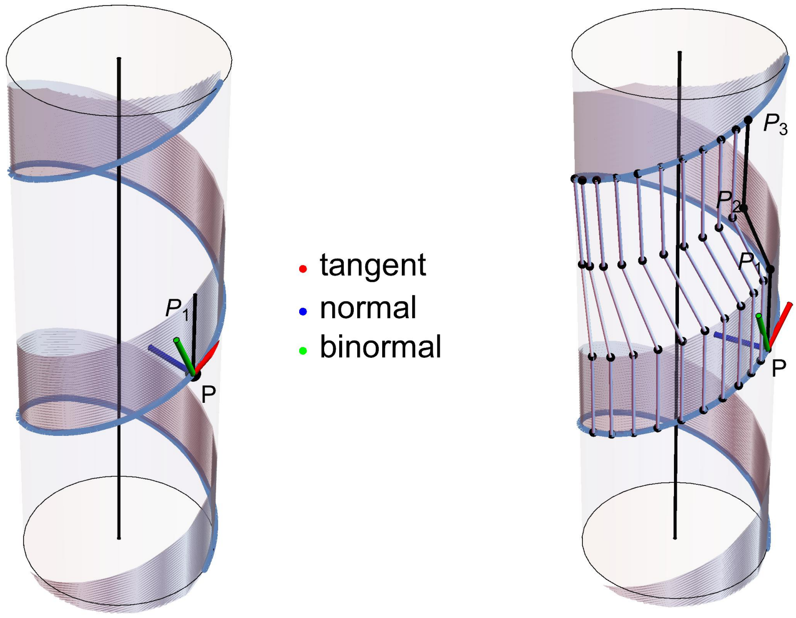

Fig. 3. Ribbons in terms of Frenet frame: as P moves along the helix , the rigid polyline $PP_1P_2P_3$ maintains fixed orientation wrt to Frenet frame. Ribbons are loci swept by this polyline.

Fig. 4. A type of twisted columns (photo Falcolini) with the corresponding models: polygonal chains integral with the Frenet frame of an helix.

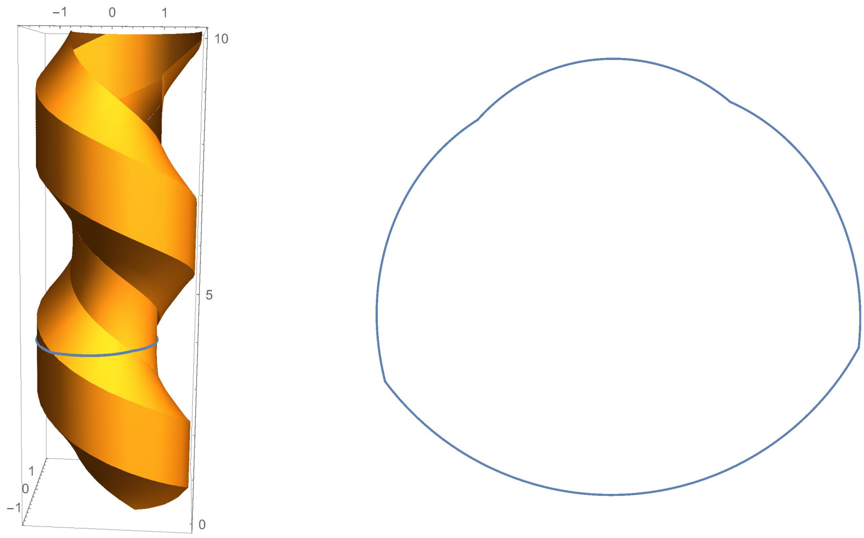

Fig. 5. Horizontal section of our compound model in mathematica.

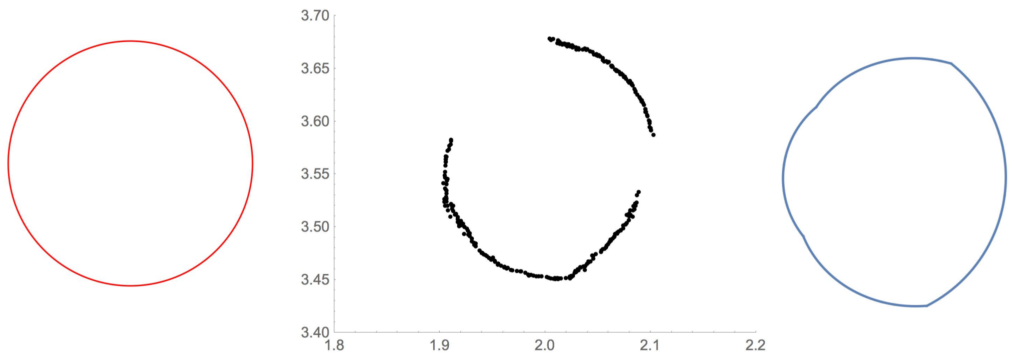

Fig. 6. Actual horizontal section of a real twisted column, obtained from the Point-cloud survey, compared with a circle and a horizontal section of our parametric compound model.

3 Summarizing: the types of medieval columns in San Paolo outside the walls

We can now describe all types of columns found in Saint Paul outside the Walls in Rome.

1. Two snakes, intertwining Fig. 7. This is the easiest column to describe mathematically, and the hardest to build in stone! Models of such columns can be found in Imperial Rome, but those were forged in bronze. Probably these medieval ones take the Roman as models, with the virtuoso technique of the sculptor. There are two identical abstract reference helices, shifted by π. Again the column can be described as a two circles of fixed radius whose center moves along the reference helices, but the circles now lie on a plane orthogonal to the tangent vector t of the helix.

2. Locally euclidean columns: the column is a cylinder. The column is ornamented with colored mosaic ribbons.

Spiraliform : Identical helices run over the cylinder, separating ornamented ribbons. Each ribbon is a strip over a cylinder, much as in the cochlear columns of Roman tradition.

Fig. 7. Marble columns sculpted as cast bronze (photo Falcolini), and its parametric model.

Vertical: the ribbons are separated by lines parallel to the generatrix of the cylinder

Horizontal: curves, piecewise helicoidal. The pieces are arranged to obtain a closed curve.

3. we notice that none of the columns we reviewed in Saint Paul's (or also in Saint John in Lateran) is of the type of later treatises, modelled by a disk, horizontal wrt to the axis of the column, and whose center moves along the elix.

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