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Università di Roma Tre, 1995-2005 Architecture and Mathematics

Andrea Pagano and Laura Tedeschini Lalli present a brief note on what has to be considered a teaching success story. They tell of ten years of teaching mathematics at the Faculty of Architecture of the Università di Roma Tre, describing some new course content that has been introduced, the methods used and, above all, the spirit that has driven the ideas about teaching mathematics to future architects. The experience covers the full curriculum of the student: from first year courses to individual projects.

Introduction

This is a success story. Its success is due, as so often is the case, to a series of fortuitous situations, and at the same time, to carefully studied choices; substantially it is due to a series of meetings with people in places that were fertile grounds for success.

It took place at the Faculty of Architecture of the Università di Roma Tre. It was the first years of the new Roman university's existence, and there was ample room for new stimuli. The farsightedness of the then-dean of the faculty, Carlo Melograni, carried on by the present dean, Francesco Cellini, led to the renewal of the content and structure of the programs. It is fundamental during such experiments that the directors are patient and inspire curiosity, enthusiasm and trust on the part of students.

Course content for the duration of the curriculum

We began with a very precise idea: that the mathematics being taught is antiquated. It isn't obsolete, but is certainly out of date. This gave us food for thought as to content: mathematics is a living subject, and it transforms itself, something that few non-mathematicians are aware of. Further, it doesn't grow through falsification of previous theories, which, happily, we continue to use in all of their rigour, but through amplification of theories or by changes in the themes being dealt with, often through the influence of other fields, of the surrounding culture, and the "spirit of the times", of which mathematics has always been an integral part.

We sought, therefore, to update the topics, the methods, and the themes, noting how in each case we continued to use the mathematics that came before as well, which typically over time acquired a kind of nimbleness, and with that a rigour and elegance which architecture students were as sensitive to as the math students were.

From the beginning, buoyed by teaching experience gained in the United States, and from our Italian background, we had some clear ideas. The first thing to be taught is a working method: the first of the tools that have to be offered to the students is the ability to read a mathematical text. We therefore tried to furnish, within the context of a typical calculus program, adequate skills for becoming aware that the ideas that were not discussed in the course exist in the literature and are accessible at various levels, even for beginners. Even during a mathematics course an ordinary student should be able to go into a scientific library and find the texts that he or she needs. This is obvious for students of the humanities, but much less so for students of so-called "service" mathematics courses. This is not even an objective of graduate courses in mathematics, and many warned us that we would be unable to achieve our goal. As it turned out, however, we were not only able to achieve our goal but came up with a sure way to overcome "math anxiety", by addressing that very subject on a personal level, and making choices that were individual as well based on available texts and interests.

First year

We pursued this objective from the very first year of the program. Throughout the entire program of mathematics in the architecture faculty, the students are encouraged to work on their own. For the first year, we provide a bibliography of topics that are not dealt with in the courses, which they are to study independently. To this end, for example, Courant and Robbins's *What is Mathematics* [1978] works wonderfully, as its rigorous arguments are often filled out by individual proofs, and the subjects covered convey very well the historical evolution of mathematics, to which architecture students are very sensitive. Especially during the courses of the first year, we suggest taking one of the problems of minima and studying it in detail.

In the same spirit, the little books of the "Little Mathematics Library" of the publisher MIR are very effective. These are collections of lectures on advanced mathematical themes that were given by scholars in high schools of the former Soviet Union. These have been translated in English and French but they can't be found anywhere; we very much hope that they will be republished.

In contrast, during the courses of the years that follow, students are invited to formalize their own cognitive journeys mathematically, drawing inspiration from sources they are most attracted to. We shall talk more about this below. It was essential for us that the students were invited to talk about mathematics among themselves, to be autonomous in their evaluations of the rigour of what they were doing, and to experience the excitement of the rigour and elegance, to measure all of this as they improved their ability to communicate with others and among themselves during the development of the ideas.

The instructor has to be able to monitor continuously the student's progress: thus the decision to give more tests during the year and to dedicate at least one hour a week to quizzes and exercises in the classroom. In particular, we open the course with a "self-test" that is fundamental in allowing the students to understand their degree of preparation. In the space of a few weeks, we give the first written test of the course, which will have little impact on the final grade but which allows the student to evaluate both the prerequisites as well as his ability to catch up on technical language and methods.

One of the main problems in the beginnings of a program in mathematics for architecture, but of this is probably true as well for "service" mathematics programs is posed by the varying degrees of student preparation. These students do not have a strong motivation towards mathematics to allow them to fill the gaps in their knowledge of the material, and very often it is necessary to resort to "courses" of a few lessons dedicated to a generic refreshing of their knowledge of the prerequisites. We have found it to be of greater advantage to present clearly to the students what the prerequisites are (few, clear and self-evaluating) and to await the first test of the course to understand the situation.

The way to proceed, therefore, is to render the students genuinely autonomous, both as regards the gaps in their knowledge as well as their individual aspirations, directing them and giving them as much guidance as possible as the case permits.

In the space of the first year we cover the contents of a university course in calculus, in one variable. Particular care is taken in this case to present all the subjects at the same time, in the same lesson, from the points of view of modelling, number, geometry and analysis. Mathematics provide bridges to act on these points of view, and to translate one into another.

The program for the first year was renovated in collaboration with Corrado Falcolini, who joined us in 2000.

From the very beginning, the results of all these things together were positive beyond even what we had hoped. In Fig. 1 we show part of the general scheme put together by Sara Bertucci, the didactic secretary of the faculty for monitoring the flow of students through the courses. From the scheme it is clear that from the year in which we began to teach the first-year course (1994-1995), not only did the students begin to pass their mathematics exams, but perhaps of greater interest, they gained confidence and trust in their ability to acquire technical language and follow scientific reasoning, and gradually they passed exams in Statics and Science of Construction that had previously been seen as obstacles in all Italian schools of architecture. Consultation with our colleagues in Statics and Science, and their continual support of our choices, allowed us to move forward and refine our objectives.

Subject	92/93	93/94	94/95	95/96	96/97	97/98	98/99	99/00	00/01
Math I	45	67	117	204	192	296	275	120	279
Math II	******	33	44	91	73	185	169	507	256
Statics	******	9	52	65	220	236	155	221	165
Science of Constr.	******	******	6	18	57	124	194	292	276

Fig. 1. Scheme for monitoring the flow of students through the courses

Second year

Still more revealing were the interventions of the second year courses (see [Pagano and Tedeschini Lalli 2005]). The basic idea was to use, in order to talk about modern mathematics, a tool that certainly is far from foreign to architecture students: the ability to visualize abstract objects, selecting the models for representation as the case requires. We think, along with the historian of mathematics Lucio Russo, that a "mathematical proof is a line of reasoning with only one conclusion." Thus, why not allow students to use their models as a starting point in order to arrive at the comprehension of fundamental questions relating to non-banal mathematical topics, such as non-Euclidean geometry?

The theory of locally Euclidean geometries is complete and elegant, and by means of it one can grapple with various problems: it provides a key to the gates of other conceptual gardens [Nikulin and Shafarevich 1987]. Let us summarize the characteristics of the theory that are striking to us, from the point of view of communications in contexts that are not mathematical as well:

- It is part of the mathematical culture of the twentieth century: too often problems deal with in university courses are limited to contributions previous to this;
- It deals with two-dimensional geometry (surfaces), the immersion of which in three-dimensional space (and possible construction of models) is not banal: we are all used to two-dimensional representations of objects in three-dimensional space; this is a new one [Lyusternik 1983], or at least little talked about;

- For mathematicians, the expression “in appropriate spaces” is so common that it is obvious, a topos of their seminars, but for non-mathematicians it is so new that it seems like science fiction. Locally Euclidean geometries provide opportune spaces. In our case, the “appropriateness” is because it provides the spaces for motor actions for some simple dynamics, such as the pendulum, while at the same time making clear possible regularities or irregularities of the trajectories, a theme that is dear to the twentieth century [Arnold 1979];
- In order to comprehend, only minimum background information (as rigorous as inevitable, however) is necessary: relations of equivalences and groups [Armstrong 1988];
- Last but not least, the contents of the theory rely at least initially on a figurative intuition, even for mathematicians; the passage from intuition to visualization and the reproduction of some characteristics, moves necessarily through a process of abstraction; at this level the formal mathematical treatment is inserted. The inverse process, from the mathematical treatment to intuition, arrives in places where intuition alone could have arrived without the math, especially as regards imaginary space. One has only to think of non-orientable surfaces in three-dimensional space (the Möbius strip and Klein bottle), whose two-dimensional representation in found in this rigorous context.

Fig. 2. Elisa Conversano, 2:3 knot on a torus (rendering by POV Ray)

Input from mathematicians of different backgrounds (analysis, mathematical physics, geometery) has permitted us to put into effect various ways of interpreting the same mathematical objects, and to use them in differing contexts.

Our courses always began with our putting the students in front of “constructable” models of mathematical objects. These models were used to deduce the essential mathematical properties, which were then appropriately formalized. Finally the process was completed in the exact opposite way, when, trusting in the nexus between visual intuition, plastic representation and formal representation, the students pushed their intuition towards shores where they hadn’t gone before, and where they couldn’t have arrived without mathematical reasoning, such as in the construction of non-orientable surfaces.

In this way we presented rigorously concepts and tools such as relations of equivalence and their quotient spaces, groups and symmetries, tilings and tangent space.

For examinations, once students passed a written test—the same for all—on the contents of the calculation of more than one variable and spatial geometry, they were asked to carry out individual project on their own, drawing inspiration from their own tastes and interests in order to compose mathematical pieces. Fig. 3 shows the work of A. Salvatore, which was completed the next year by A. Carlini and A. Spatafora, who, using groups for tiling the plane, reconstructed the original appearance of the pavement of the tabernae of Trajan's Market in Rome.

We took advantage of seminars and mini-courses held by professors from other institutions about their research during the second year courses. The first experience was with Capi Corrales Rodrigañez (University Complutense of Madrid), who experimented first with us a lecture [Corrales Rodrigañez 2000a] that later became a book [Corrales Rodrigañez 2000b]. After Corrales's course there were many independent student projects, part of which were published on a CD-ROM entitled "Dalla scatola alla rete in matematica e pittura" (From the box to the grid in mathematics and painting), edited by Giulia Longo; others appeared in the exhibit entitled "Spazi matematici, spazi pittorici" (Mathematical spaces, painterly spaces), organized by the architect Michele Furnari in our faculty in December 1998.

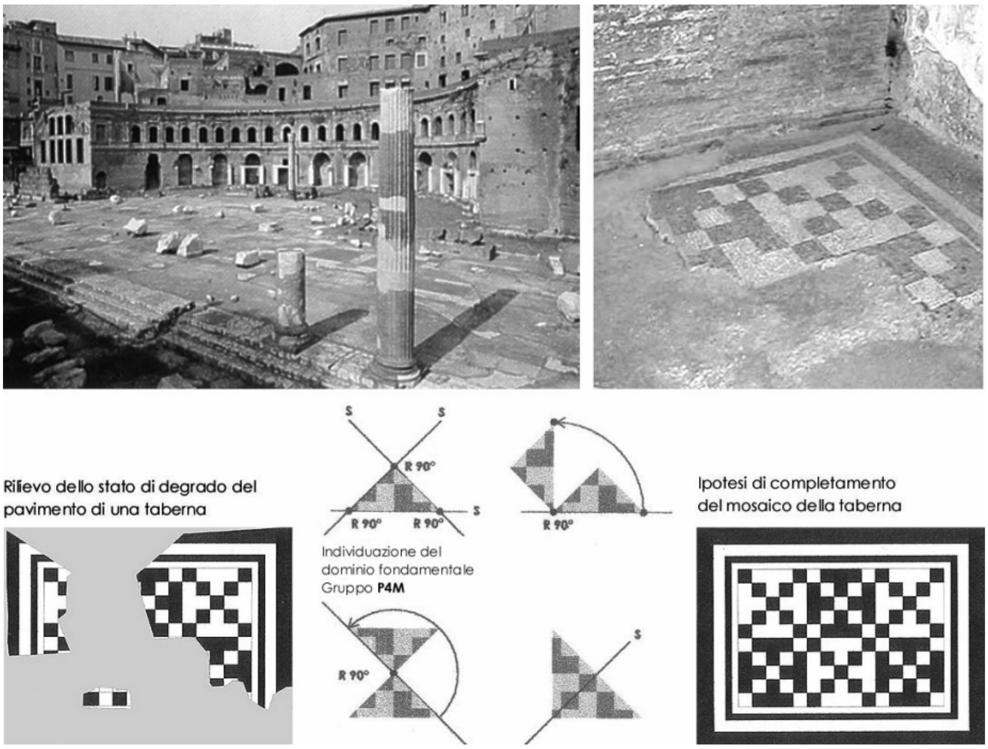

Fig. 3. A. Carlini, A. Spatafora, reconstruction of the pavement of the tabernae of Trajan's Market, Rome

Fig. 4 and Fig. 5 show works of abstract spaces, from which it is pretty clear what we mean when we say that we let the students draw inspiration where they may in order to make mathematics. In the case where the mathematics in question was made by others, this means learning.

Among the other scholars who participated, there was A. Toschi, who was the mathematics consultant for the painter Paola Levi Montalcini in the 1970s, when she was most interested in certain curves and their composition, which were realized on copper (Fig. 5).

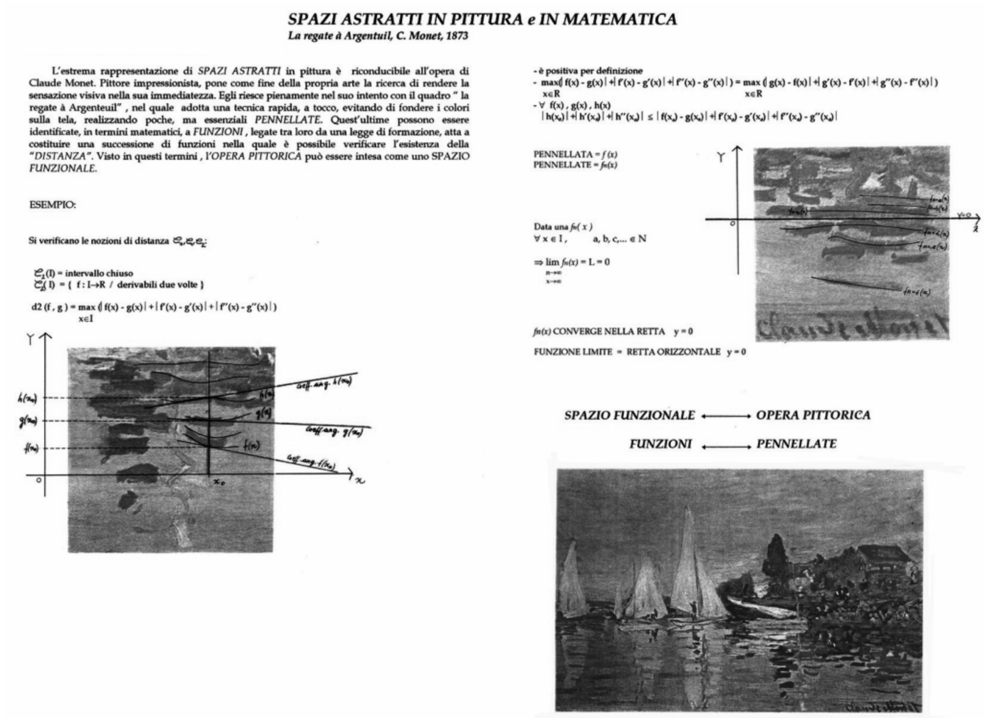

Fig. 4. R. Arringhini, M. Chialastri, V. Nuccitelli. From the final exam of the course "Spazi astratti in mathematica e in pittura" (Abstract spaces in mathematics and painting)

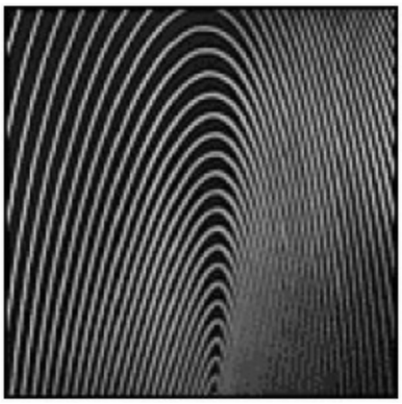

Fig. 5. Paolo Levi Montalcini, Whitney

Elective courses

The courses we have described up to now were required courses for all students. We also had an elective fifth-year course, taught in collaboration with Gian Marco Todesco. Half of the course had a mathematical content, while the other half took place in the computer lab of the faculty.

We nicknamed this course "Mathematics as a palette". The design of forms, with both straightedge and compass as well as a sophisticated CAD system, presupposes an underlying mathematical structure (perhaps one that is hidden behind the intuitiveness of the instrument). The course offers students the possibility of experimenting in the lab some techniques that are relatively non-conventional in an environment that is very versatile and powerful. The object is to improve comprehension of existing instruments and to attempt the design of some entirely new ones. In contemporary architecture these design techniques play an increasingly important role.

At the end of the course the students will have an knowledge that is both mathematical and operative of the following topics:

- Regular and uniform polyhedra, polyhedra and polyhedral symmetry;
- Geodesic domes and networks;
- Design procedures, cycles and other control structures, the generation of recursive structures (fractals), definition of mobile structures;
- Parametric curves and parametric surfaces, how to select the most suitable formula in order to represent a given shape;
- Conch shells, helixes, serrated surfaces, topologically complex surfaces;
- Definite surfaces in implicit form;
- Some Renaissance design tools (Leonardo's elliptograph, etc.);
- Trihedron of vectors attached to a curve and its use in animation;
- Fractal objects and their fractal dimensions.

The environment of development is a program that allows one the generation of a photorealistic image starting with a formal description of the scene. The program is called POV Ray, and is available as shareware on the Internet. It is relatively easy to learn. This is an enormous value, also where teaching is concerned: the availability of the program in the public domain permits the student to be immediately immersed in a group of people who develop software.

Projects

Hobermann's sphere. A study of Hobermann's sphere made by Riccardo Pejani starting with a toy sold by street vendors in Rome. The sphere was reconstructed, the geometry of the hinges that permit the enlargement or reduction of the sphere was studied in animation. Taking this project as a starting point, there is now in progress a master's thesis on the structure of foldable roofs.

Egg. This was one of the most instructive projects. It started around Easter, when chocolate eggs are given as gifts. Gian Marco Todesco left the classroom with these instructions, "Design an egg," and many models were produced, each different. Alban Hintzy collected them, verified the nearness of their parameters to those of a hen's egg, and then did a study of the efficiency of the various methods. The methods range from parametric modification of three classic methods for drawing ellipses, to the techniques that are most used by architects today. The study of the efficiency and economy of the various designs (machine time, number of parameters to fix) is therefore essential.

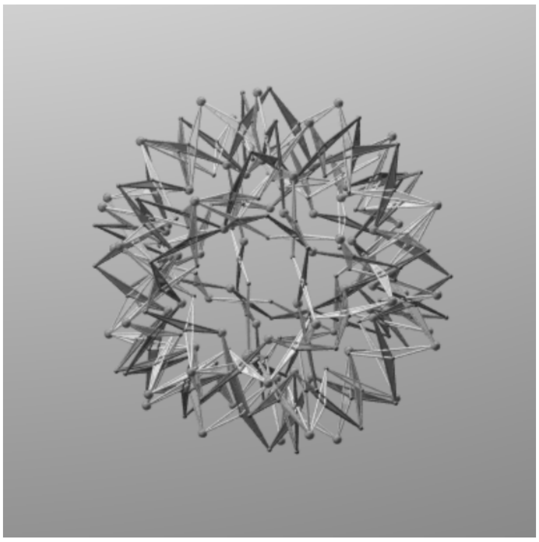

Fig. 6. R. Pejani, study for the reconstruction of the movement of Hobermann's sphere.
Rendering with POV Ray

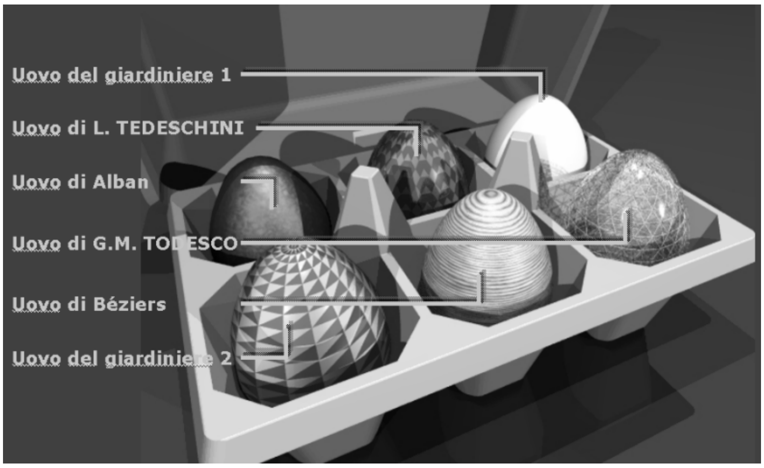

Fig. 7. Six methods for designing an egg

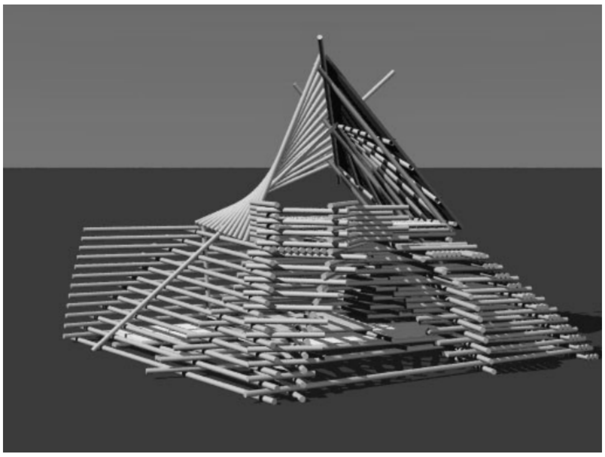

Fig. 8. M. Ferrelli, study of light phases for a meditation structure

To conclude, from the many works in progress, we want to present that which is most closely related to architecture, in the sense that the student, M. Ferrelli, had previously designed this "meditation place", building a model. The use of the computer permitted the expansion into a more complete project, in order to understand the deeper underlying structure and to weigh the adaptability and orientability according to the site conditions. The structure is equipped with a roof that functions as a solar meridian.

Translated from the Italian by Kim Williams

References

ARMSTRONG, M.A. 1988. *Groups and Symmetry*. New York: Springer-Verlag.

ARNOLD, V.I. 1979. *Metodi matematici in meccanica classica*. City: Editori Riuniti.

CORRALES RODRIGAÑEZ, C. 2000a. Dallo spazio come contenitore allo spazio come rete. Pp. 123-138 in *Matematica e cultura 2000*, M. Emmer, ed. Milan: Springer-Italia.

———. 2000b. *Contando el espacio*. Madrid: Mobcoop Edicciones.

COURANT R. and H. ROBBINS. 1978. *What is Mathematics? An elementary approach to ideas and methods*. Oxford: Oxford University Press.

LYUSTERNIK, L.A. 1983. *The Shortest Lines.* Little Mathematics Library. Moscow: MIR Publishers.

NIKULIN, V.V. and I.R. SHAFAREVICH. 1987. *Geometry and Groups*. M. Reid, trans. Berlin and New York: Springer-Verlag.

PAGANO, A. and L. TEDESCHINI LALLI. 2005. *Geometrie e Modelli.* Rome: Aracne.

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