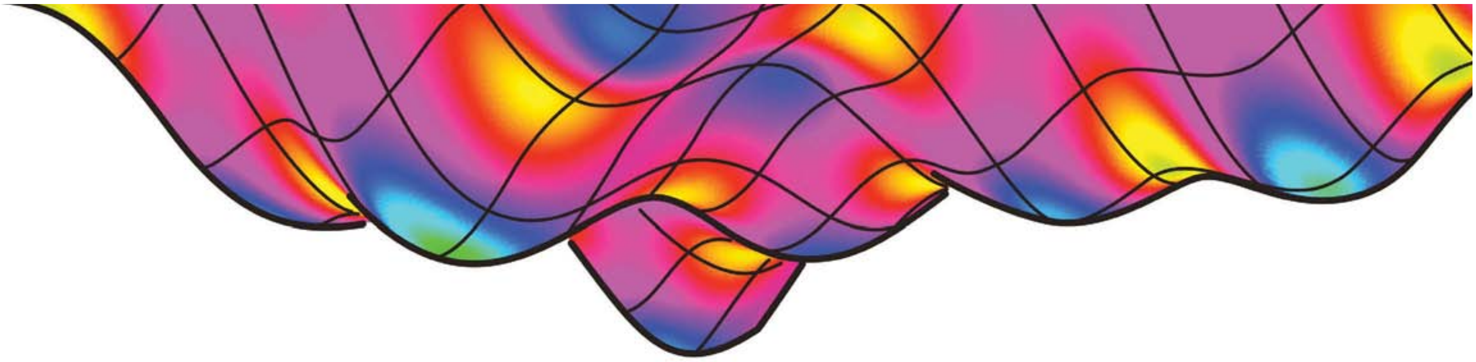


Curvature of Surfaces

Curves revisited. To compute the tangent of a curve, we need a first derivative of its mathematical representation. Recall from Chapter 7 what we have said about curvature and the osculating circle of a planar curve c. The computation of these entities requires derivatives up to second order, but not higher than that. This implies the following fact: There are many curves d that touch the given curve c at a chosen point p and have the same curvature k and osculating circle there.

We say that any of these curves *d osculates c at p*. Of course, the osculating circle itself is one example, but there are infinitely many osculating curves. An easy way to obtain an osculating parabola is based on Taylor's theorem (Figure 14.3). Let the curve c be given as the graph of a function $f(x)$; that is, the curve has the form $y = f(x)$. To pick a point on this curve, we choose an x value (e.g., $x = a$) and obtain the curve point $p = (a, f(a))$. We also compute the first and second derivative of f evaluated at $x = a$ and denote these values $f'(a)$ and $f''(a)$, respectively. We then consider the following function

$$g(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2} f''(a)(x - a)^2. \qquad (14.1)$$

If we insert $x = a$, we obtain $g(a) = f(a)$. This says that the curve d given by $y = g(x)$ passes through the point $p = (a, f(a))$. To compute the first derivative of g, we note that $a, f(a), f'(a)$, and $f''(a)$ are constants (do not depend on x). Thus,

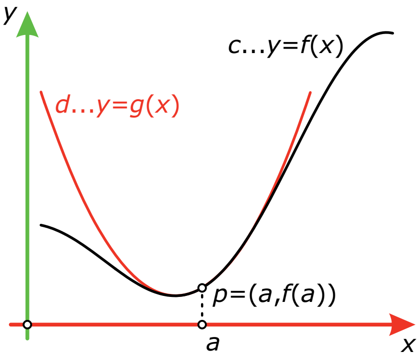
$$g'(x) = f'(a) + f''(a)(x - a).$$

We see that $g'(a) = f'(a)$, which means that the curve d and our original curve c have the same tangent at p. The equation of the tangent is $y = f(a) + f'(a)(x - a)$. Curves c and d even have the same curvature at p! To prove this, we differentiate again as

$$g''(x) = f''(a).$$

Thus, the second derivative of g is constant and is the same as the second derivative of f at $x = a$. Because functions f and g agree at $x = a$ in all derivatives up to second order, their graph curves c and d osculate at p. The function g is a quadratic function of x and is called the second-order Taylor approximation of f at $x = a$. As the graph of a quadratic function, the curve d is a *parabola*. It is important to note that the second derivative $f''(a)$ is in general *not* the curvature at p. However, the following is a special case: If $f''(a) = 0$, the point p is an inflection point and the curve d is just the tangent of c at p.

Fig. 14.3
Taylor's theorem allows us to easily compute an osculating parabola of a curve given as graph of the function $y = f(x)$.



Example:

Curvatures of a sine curve. We let $f(x) = \sin x$; that is, we study the graph curve c of the sine function (Figure 14.4). Note that $f'(x) = \cos x$, $f''(x) = -\sin x$.

- Let's start with $a = 0$. Here, we have $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$. Inserting this into the second-order Taylor expansion g from Equation 14.1, we see that the osculating parabola $y = g(x)$ has the equation $y = 0 + 1 \cdot x + ½ \cdot 0 \cdot x^2$; that is, it agrees with the tangent $y = x$ in accordance with the fact that the point (0,0) is an inflection point of the sine curve. Inflection points also occur for $a = \pi$, $2\pi, \ldots, -\pi, -2\pi, \ldots$
- Next, we set $a = \pi/2$ and note $f(\pi/2) = 1$, $f'(\pi/2) = 0$, $f''(\pi/2) = -1$. This yields $g(x) = 1 + 0 \cdot (x - \pi/2) - ½ \cdot 1 \cdot (x - \pi/2)^2$, and thus we obtain the osculating parabola $y = 1 - ½ \cdot (x - \pi/2)^2$. Its vertex is the considered curve point $(\pi/2, 1)$, and its curvature at this point is -1 (see discussion following).
- Finally, we consider $a = -\pi/4$, which leads to $f(-\pi/4) = -\sqrt{2}/2$, $f'(-\pi/4) = \sqrt{2}/2$, $f''(-\pi/4) = \sqrt{2}/2$. This yields $y = (\sqrt{2}/2) \cdot [-1 + x + \pi/4 + ½ \cdot (x + \pi/4)^2]$ as the equation of the osculating parabola.

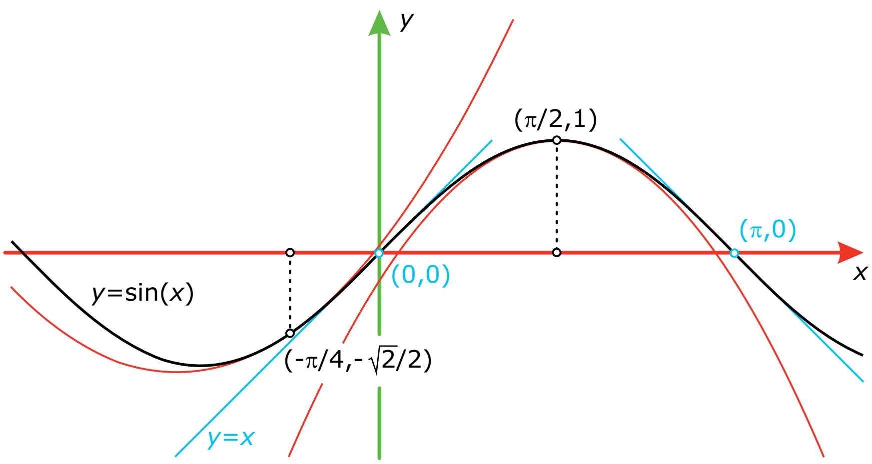


Fig. 14.4
Osculating parabolas for the curve $y = \sin x$. We focus on a few points of this curve: $(x,y) = (\pi/2, 1)$ describes a point p with vanishing first derivative (and there p is the vertex of the parabola); $(-\pi/4, -\sqrt{2}/2)$ is a "general" point; and (0,0) and $(\pi,0)$ are inflection points where the osculating parabola degenerates to the tangent.

We may take a special parabola, $y = \frac{1}{2} x^2$, and use the formula of Chapter 7 to compute its curvature k at the origin $(x,y) = (0,0)$, which is the vertex of this parabola (Figure 14.5). We find $k = 1$; that is, the radius r of the osculating circle at the vertex is also $r = 1$. This is in agreement with the fact that the radius of the osculating circle at the parabola's vertex equals twice the distance of the focal point to the vertex and this focal distance is ½.

In an analogous way, the curvature of the parabola $y = (k/2)x^2$ for some constant value k at the origin is equal to k. Note that k is also the second derivative of the function $g(x) = (k/2)x^2$. Therefore, in this special case the second derivative gives us the right value of the curvature at the origin. The reason behind this is the vanishing first derivative at the origin.

The osculating paraboloid. Now we are well prepared for a discussion of the curvature behavior of smooth surfaces. Inspired by the results on curves, we will use Taylor's theorem—but now for two variables, x and y. We do not even quote its general form, but just outline the considerations that lead to a simple result upon which the further discussion can be based.

The goal is to obtain a counterpart of an osculating parabola, but already in a special position, analogous with Figure 14.5. Hence, we select a point p of the surface S (where we want to study the curvature) and let p be the origin of the coordinate system. Moreover, we place our coordinate system such that the xy-plane (equation $z = 0$) is the tangent plane of S at p. Our osculating surface P at p, which is called the *osculating paraboloid of S at p*, now has an equation of the form

$$z = ax^2 + bxy + cy^2.$$

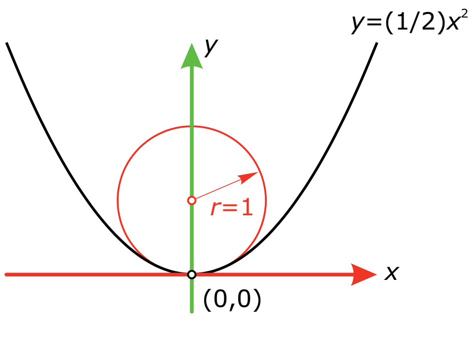
The values of a,b,c are certain second-order derivatives, but are not needed right now. This surface P is in general a paraboloid. In special cases, it is a parabolic cylinder. Alternatively, if $a = b = c = 0$, it is just the plane $z = 0$. Because all of these surfaces have two orthogonal planes of symmetry, we may further adapt our coordinate system and make sure that the xz-plane and the yz-plane are the symmetry planes. This removes the term bxy in the previous equation. We denote the resulting coefficients of x^2 and y^2 as $k_1/2$ and $k_2/2$, respectively. Thus, our surface P (which has the same curvature behavior as S at p) has the simple equation

$$z = (k_1/2)x^2 + (k_2/2)y^2. \tag{14.2}$$

A complete derivation of this result would require more mathematics. For our purposes, it is sufficient to know that this equation contains the curvature behavior of a general smooth surface S where smoothness means that S has a twice-differentiable mathematical representation.

Note the relations of Equation 14.2 to the curve case. In the xz-plane (equation $y = 0$) we have the parabola p_1: $z = (k_1/2)x^2$ with curvature k_1 at the origin p. The intersection curve of P with the yz-plane (equation $x = 0$) is the parabola p_2: $z = (k_2/2)y^2$ with curvature k_2 at p. These curvatures k_1 and k_2 are called *principal curvatures* of P and S at p. The x- and y-axes, which we have chosen based on geometric considerations, are called *principal directions* at p.

Fig. 14.5
The curvature of the parabola $y = (1/2)x^2$ at its vertex (origin) is 1. More generally, the curvature of the parabola $y = (k/2)x^2$ at its vertex (0,0) is k. This is the constant value of the second derivative of the function $g(x) = (k/2)x^2$.



Normal curvatures. Normal curvatures of a surface S at a point p are obtained as follows. We intersect S with a plane R through the normal n of S at p and measure the curvature k_n of the resulting intersection curve at the point p. There are infinitely many normal curvatures, depending on which plane R we have chosen. We know from the previous discussion that we may use the osculating paraboloid P from Equation 14.2 instead of S.

The surface point p is then the origin and the normal at p is the z-axis. The plane R can be defined by its angle α against the x-axis (first principal direction). Introducing a coordinate system (u,z) in the plane R as shown in Figure 14.6, we have the following relations between the u coordinate of points in R and their x and y coordinates:

$$x = u \cdot \cos\alpha, y = u \cdot \sin\alpha.$$

We can insert this into Equation 14.2 and obtain for the intersection curve between P and R the equation

$$z = (1/2)[k_1(\cos\alpha)^2 + k_2(\sin\alpha)^2]u^2.$$

This is a parabola $p(\alpha)$, whose curvature at the origin is the desired normal curvature $k_n(\alpha)$ to direction angle α. Because the coefficient $(1/2)[k_1(\cos\alpha)^2 + k_2(\sin\alpha)^2]$ of u^2 equals $k_n/2$, we have the following result,

$$k_n(\alpha) = k_1(\cos\alpha)^2 + k_2(\sin\alpha)^2. \qquad (14.3)$$

Hence, knowing the principal curvatures k_1 and k_2 we can compute the normal curvature $k_n(\alpha)$ to any given direction angle α. The tangent of $p(\alpha)$ at p (u-axis in Figure 14.6; intersection line of the plane R and the tangent plane at point p) is also called the *direction* with which the normal curvature $k_n(\alpha)$ is associated.

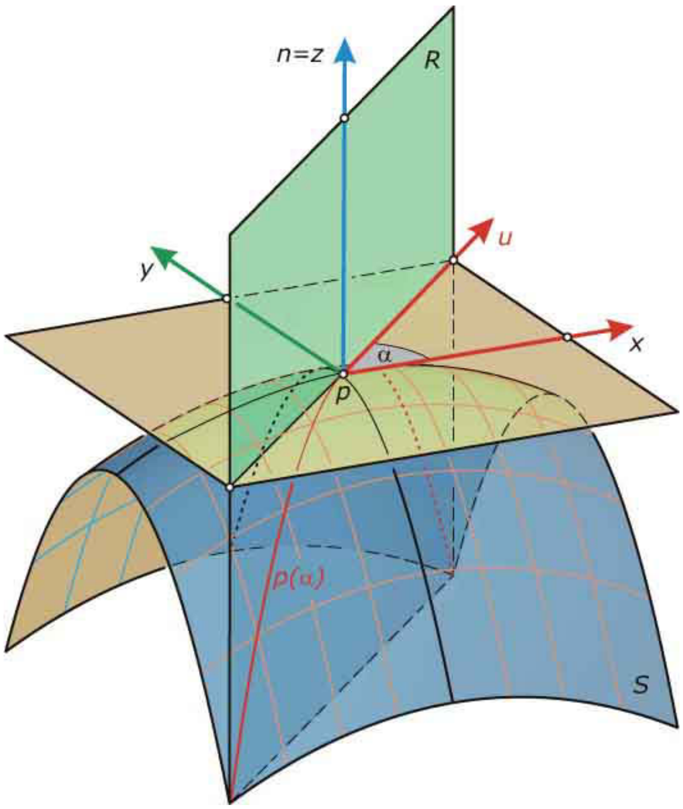


Fig. 14.6
Normal curvatures of a surface S at a point p are the curvatures of the intersection curves with planes R through the surface normal n. Using the paraboloid P from Equation 14.2, we can introduce a (u,z)-coordinate system in R and in this way obtain Euler's formula (Equation 14.3)—which relates the normal curvature to direction angle α with the two principal curvatures.

Classification of surface points. Surface points can be classified according to the type of the osculating paraboloid.

- **Elliptic surface point.** Here, the osculating paraboloid P is an *elliptic paraboloid* (Figure 14.7). Mathematically, in this case the principal curvatures k_1 and k_2 are of the same sign and different from zero. Geometrically, this implies that the parabolas p_1 and p_2 are open toward the same side. The paraboloid P and the underlying surface S lie locally on one side of the tangent plane T at the considered surface point p. A visualization via planar intersection curves is shown in Figure 14.8.
- **Hyperbolic surface point.** In this case, the osculating paraboloid P is a *hyperbolic paraboloid* (Figure 14.9). The principal curvatures k_1 and k_2 have different signs and thus the parabolas p_1 and p_2 are open toward different sides. P and the underlying surface S lie locally on both sides of the tangent plane T at p (Figure 14.10). T intersects the surface S along a curve, which has a double point at p. The tangents of this curve at the double point p are the so-called *asymptotic directions* along which the normal curvature vanishes. They are also the intersection lines between the osculating paraboloid P and T. Euler's formula (Equation 14.3) can be used to compute the angles α between principal directions and the asymptotic directions. We simply have to solve $k_n(\alpha) = 0$. Note that due to the different sign of k_1 and k_2 this equation has a solution. It does not have a solution at an elliptic point where k_1 and k_2 have the same sign.

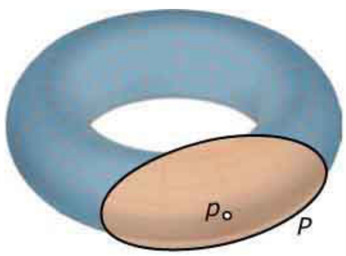


Fig. 14.7
At an elliptic surface point p, the osculating paraboloid P is an elliptic paraboloid.

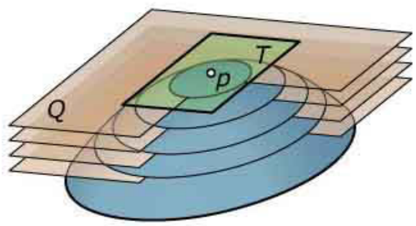


Fig. 14.8
At an elliptic surface point p, the surface S lies locally on the same side of the tangent plane T. Intersection with a plane Q that lies on this side and is parallel to T gives a closed curve whose shape approaches that of an ellipse as Q gets closer and closer to T.

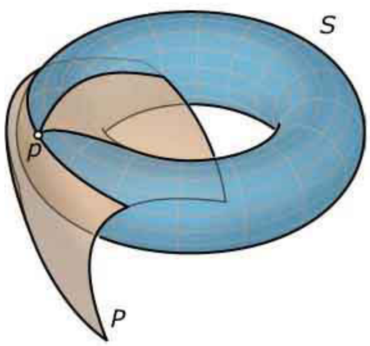


Fig. 14.9
At a hyperbolic surface point p of a surface S, the osculating paraboloid P is a hyperbolic paraboloid. Thus, the surface has locally a saddle-like shape.

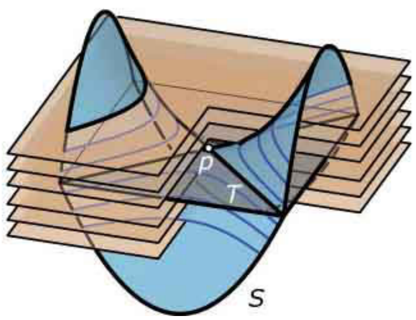


Fig. 14.10
At a hyperbolic surface point p, the surface S lies locally on both sides of the tangent plane T. The intersection curve between T and S passes through p and has a double point there. The figure also shows some intersection curves with planes parallel to T.

- **Parabolic surface point.** Here, the osculating paraboloid P is a *parabolic cylinder* (Figure 14.11). One principal curvature vanishes, and we may assume that $k_2 = 0$. The corresponding direction is the only direction with vanishing normal curvature. The local behavior with respect to the tangent plane is more complicated to explain. Some essential cases are shown in Figure 14.12. In general, parabolic points occur along curves that separate regions of elliptic points from regions with hyperbolic points.
- **Flat point**. In this case, both principal curvatures vanish and therefore all normal curvatures are zero. The osculating surface P degenerates to the tangent plane (Figure 14.13).

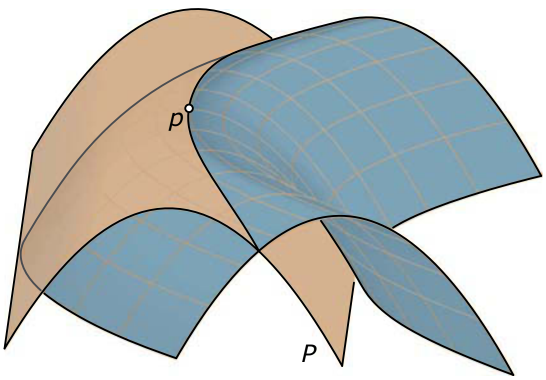


Fig. 14.11
At a parabolic surface point p, the osculating "paraboloid" P is a parabolic cylinder.

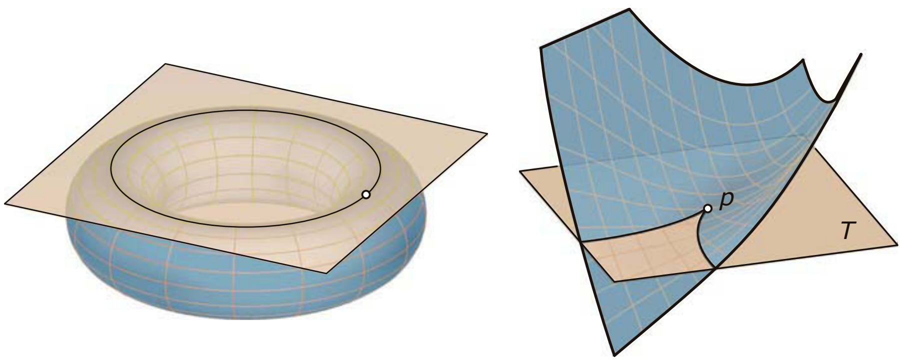


Fig. 14.12
Examples of the local behavior of a surface at a parabolic point, visualized with the help of the intersection curve between surface and tangent plane.

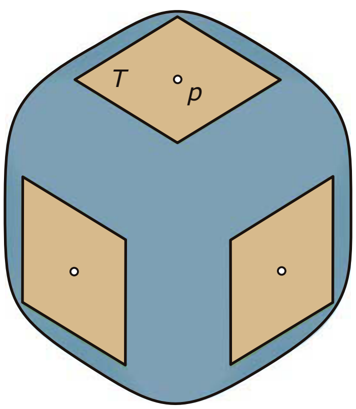


Fig. 14.13
At a flat point p, the osculating surface P is the tangent plane T. All normal curvatures at p vanish.

Example:

Surfaces of revolution. We illustrate the classification of surface points of a ring torus S (Figure 14.14). There are two planes orthogonal to the axis, each of which touches the torus along a circle. These circles contain the parabolic points. Obviously, the outer ring has only elliptic points because there the torus lies locally on the same side of the tangent plane.

Analogously, the remaining part is the one that has only hyperbolic points, where the torus lies locally on both sides of the tangent plane. At each point, the principal directions are the tangent to the circular profile and the tangent to the parallel circle (rotational path). Analogously, we can discuss the distribution of elliptic and hyperbolic points along other types of rotational surfaces. There, it is important to note that an inflection point of the profile gives rise to a parabolic surface point (Figure 14.15).

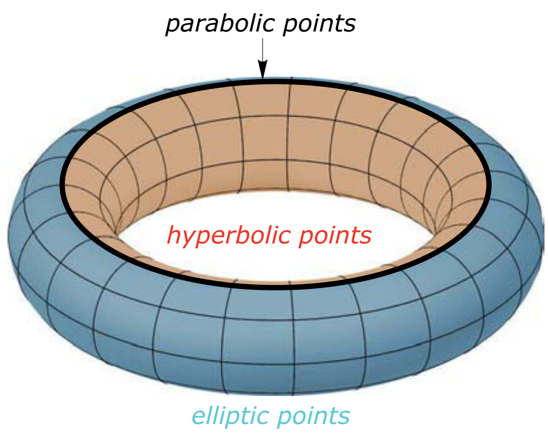


Fig. 14.14
Two circles segment a ring torus into a region with elliptic points and a region with hyperbolic points (saddle-like points). The two circles contain only parabolic surface points.

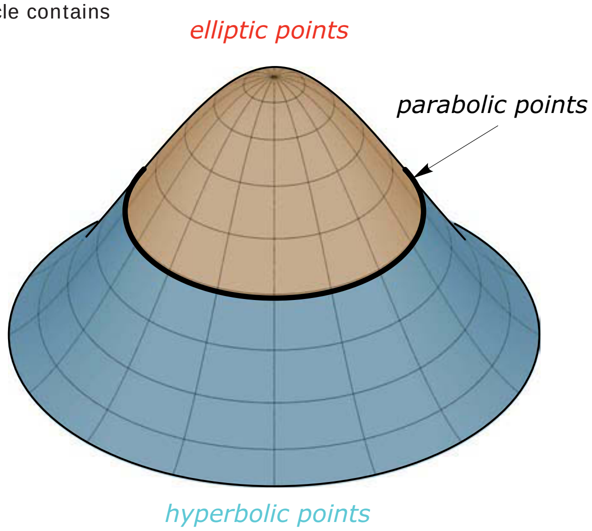


Fig. 14.15
A surface of revolution, classified into regions of elliptic and hyperbolic points. The separating circle contains the parabolic points.

Example:

Freeform surfaces. Segmentation of a surface into regions with elliptic and hyperbolic points can be done with modeling programs that have a tool for visualizing *Gaussian curvature* $K = k_1 \cdot k_2$, the product of principal curvatures. Gaussian curvature is further discussed in material following (Figure 14.16).

Here, we can already note that Gaussian curvature zero characterizes a parabolic point or flat point. At an elliptic point, both principal curvatures have the same sign and thus we have a positive Gaussian curvature. Likewise, a hyperbolic point has principal curvatures of different sign and therefore it is also characterized by negative Gaussian curvature.

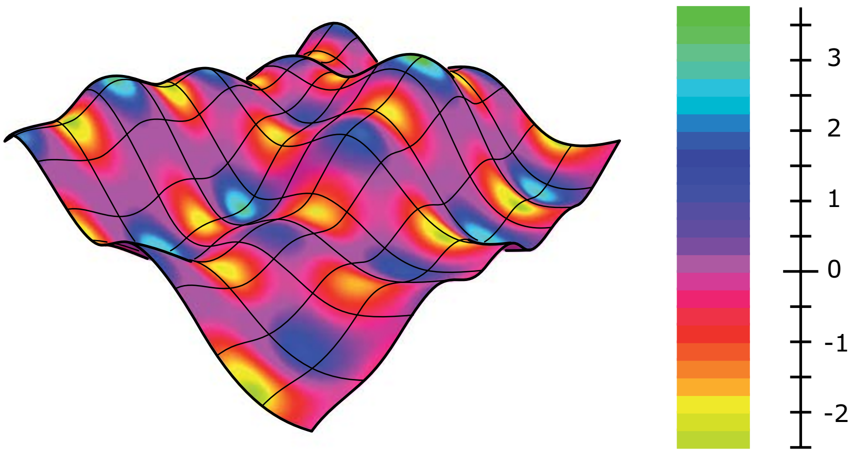


Fig. 14.16
A freeform surface with a color-based visualization of Gaussian curvature K. Positive K characterizes an elliptic point, negative K belongs to a hyperbolic point, and $K = 0$ holds at parabolic points.

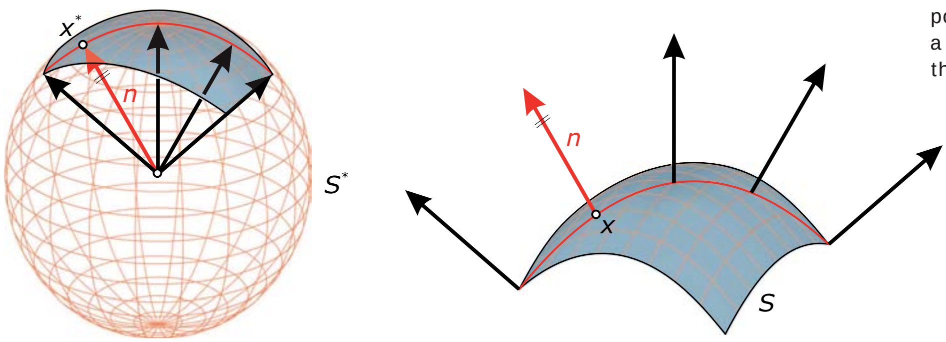


Fig. 14.17
The Gaussian spherical mapping views the outward unit normal vector at any point x of the considered surface S as a coordinate vector of a point x^* on the unit sphere S^*.

Gaussian curvature. The product $K = k_1 \cdot k_2$ of principal curvatures is called *Gaussian curvature*. Before we discuss some of the properties of Gaussian curvature, we briefly illustrate the approach taken by Carl Friedrich Gauss (1777–1855) to define this measure of surface curvature. It uses the following mapping from a surface S onto the *unit sphere* S^* (= sphere of radius 1 whose center is the origin of the underlying Cartesian coordinate system): In the neighborhood of the considered point p on S, we use a consistent orientation of the surface normals. This means that we distinguish between two sides of the surface, and call them the outer and inner side. In this way, each point x in a neighborhood of p has an outward unit normal vector (called n) that points to the outer side. This vector has length 1, and seen as coordinate vector of a point thus represents a point on the unit sphere S^*. Summarizing, we have to do the following (Figure 14.17): For a point x of S, we take the outward unit normal vector n and view it as coordinate vector of a point x^* on the unit sphere S^*. The mapping $x \rightarrow x^*$ is called *Gaussian spherical mapping*. The image of a surface S under this mapping is called its Gaussian image.

Let's consider some examples.

- If S is a *plane*, all normals are parallel and therefore any point of the plane is mapped to the same point of the Gaussian sphere. The entire Gaussian image of the plane S is this single point.
- Let S be a *sphere* of radius R (Figure 14.18). On S, we consider a disk D with spherical center p. The Gaussian mapping maps the disk D to another disk D^* on the Gaussian sphere S^*. Connecting the boundary circle k of D with the center of the sphere S, we obtain a cone of revolution N—which is also formed by the normals of S along k. The corresponding cone N^*, which connects the boundary circle k^* of D^* with the center of S^*, is congruent to N. Therefore, D results from D^* by applying a uniform scaling with factor R and a translation. Hence, the surface area A^* of D^* and the area A of D possess the ratio $A^*/A = 1/R^2$. Because the normal curvatures of the sphere R at any of its points equal $k_1 = k_2 = 1/R$, the Gaussian curvature $K = k_1 \cdot k_2$ of the sphere equals $K = 1/R^2$ and thus K agrees with the ratio A^*/A.

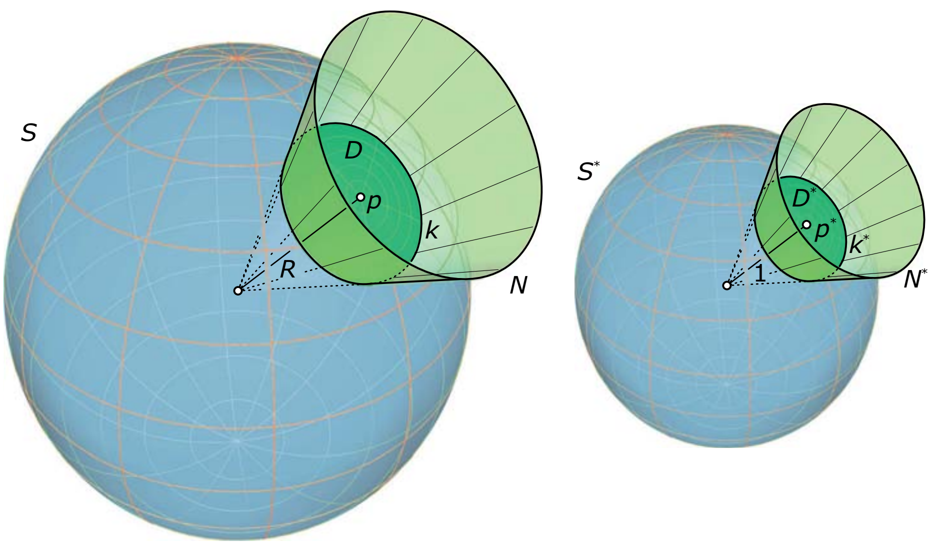


Fig. 14.18
The Gaussian spherical mapping applied to a spherical disk D. The normals at the boundary of D form a cone of revolution N, which is congruent to the corresponding cone N^* formed by the normals along the boundary circle k^* of D^*.

Let S now be an arbitrary surface, and let p be a point on it. We consider a local neighborhood D of p on S. With the Gaussian mapping, it is mapped onto a neighborhood D^* of p^* on the sphere S^* (see Figure 14.19). Obviously, if the variation of surface normals over D is strong the domain D^* will be larger than for a weak normal variation (see the examples considered previously). In other words, the ratio A^*/A of the area A^* of D^* and the area A of D will measure the variation of normals—which is clearly a measure of curvature.

Now one considers the limit of the area ratio A^*/A when D shrinks to a point p. Of course, also D^* shrinks to a point p^* and thus both areas are zero. This is no problem, because the limit of the ratio A^*/A exists (if the representation of the surface S is twice differentiable). *The limit of the ratio A^*/A is the Gaussian curvature K at p*. We have verified this previously for the very special example of a sphere, but it is true for general surfaces. We may therefore say that the *Gaussian curvature measures the local area distortion under the Gaussian spherical mapping*.

Isometric mappings and cartography. Cartography is concerned with the generation of planar maps of the earth's surface. If one approximates the surface of the earth by a sphere S, the problem is to find an appropriate mapping of a part D of S onto a planar domain D_1. Ideally, one would like to have a mapping that preserves the length of any path (river, street, and so on) on the earth—of course, up to an appropriate scaling factor that applies to all distances.

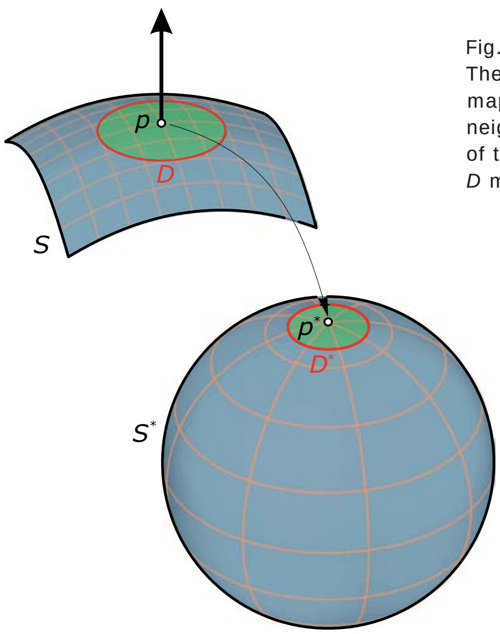
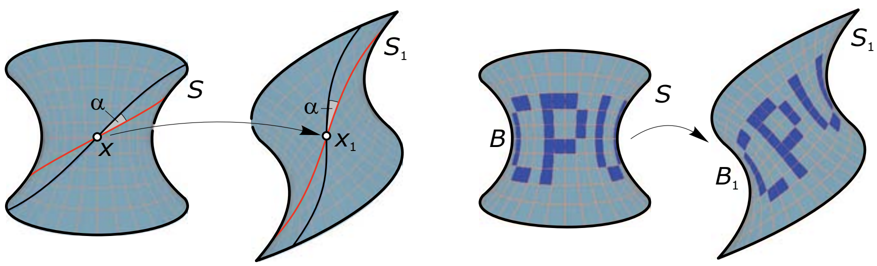


Fig. 14.19
The Gaussian spherical mapping maps a neighborhood D of p to a neighborhood D^* of p^*. The ratio A^*/A of the area A^* of D^* and the area A of D measures the variation of normals on D. This is an "averaged" measure of curvature on the domain D. The Gaussian curvature K is the limit of A^*/A when D shrinks to the point p.

Fig. 14.20
An isometric mapping between two surfaces preserves the lengths of curves and the intersection angles between curves. The Gaussian curvature at corresponding points x and x_1 are equal. Moreover, the surface area of any domain B and its image domain B_1 are the same.



In this way, one would have the ideal impression of the distances between different places. However, such a mapping does not exist! A simple proof can be based on the fact that a circle c (spherical radius r) on S needed to be mapped onto a circle c_1 of radius r in the plane, but these circles have different length. Another proof follows from a much more general result by C. F. Gauss, which concerns so-called *isometric mappings* between two surfaces S and S_1 (see Figure 14.20).

Such a mapping maps any point x of S to a point x_1 of S_1. A curve c on S is mapped to a curve c_1 on S_1 such that c and c_1 have the same length. Hence, distances measured along curves are preserved. C. F. Gauss proved that *Gaussian curvature is preserved under isometric mappings*. If S has Gaussian curvature K at point x, S_1 must have the same Gaussian curvature K at the image point x_1 of x. Because the sphere has Gaussian curvature $1/R^2$ and the plane has Gaussian curvature zero, there is no way to find an isometric mapping between the sphere and the plane. Hence, *there is no distortion-free map of the earth*.

One can show that an isometric mapping $S \rightarrow S_1$ also *preserves* the intersection *angles* between curves (see Figure 14.20). Moreover, it *preserves surface areas*: a domain B in S and its corresponding domain B_1 in S_1 have the same surface area.

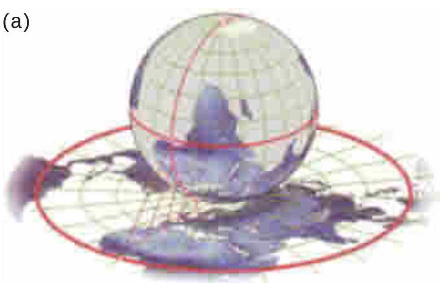
However, an angle-preserving mapping (also called *conformal mapping*) need not be an isometric one. In fact, there are many angle-preserving maps between any pair of surfaces. Let's compare this result with our discussion of the inversion in Chapter 13. If an inversion maps a surface S to a surface S_1, the mapping between the two surfaces S and S_1 generated in this way is conformal.

However, there are other ways of obtaining a conformal mapping between two surfaces S and S_1—ways that are actually related to the fact that there are many conformal mappings of the plane onto itself (see Chapter 5). Requiring angle preservation for a mapping of thre -dimensional space onto itself (not just between two surfaces) is much more restrictive and only leads to similarities, inversions, and their combinations (see Chapter 13).

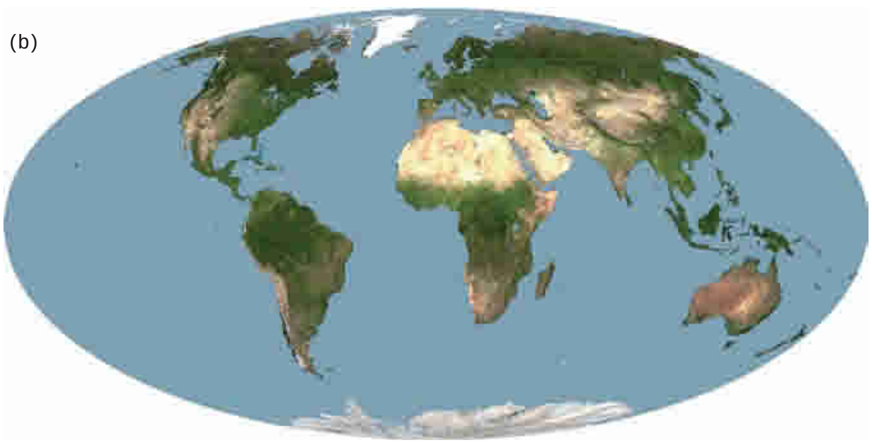
Likewise, there are many area-preserving mappings. Frequently used mappings of the earth are either angle preserving or area preserving (see Figure 14.21). However, they cannot have both properties because preservation of both angles and areas implies an isometric mapping.

Fig. 14.21
Examples of mappings used in cartography.
(a) The angle-preserving stereographic projection (see Chapter 2)
(b) An area-preserving mapping attributed to K. B. Mollweide (1805).

(a)



(b)



Developable surfaces. A surface S that can be mapped into the plane by an isometric mapping is called a *developable surface*, and the isometric planar image is called its *development*. Due to the preservation of Gaussian curvature under isometric mappings, a developable surface must have *vanishing Gaussian curvature* $K = 0$ at all of its points. Thus, these surfaces are also called *single curved surfaces* (in contrast to double curved surfaces with K different from zero; see Figure 14.22). We will study these surfaces in Chapter 15. Here, we only give two examples: cylinder surfaces and cones are developable surfaces.

Mean curvature. Another important measure of surface curvature is *mean curvature* $H = (k_1 + k_2)/2$, the arithmetic mean of the principal curvatures. Surfaces with vanishing mean curvature are *minimal surfaces*, which appear (for example) as shapes of soap films through a closed wire (Figure 14.23). These and related equilibrium shapes of surfaces are discussed in Chapter 18 in connection with shape optimization problems.

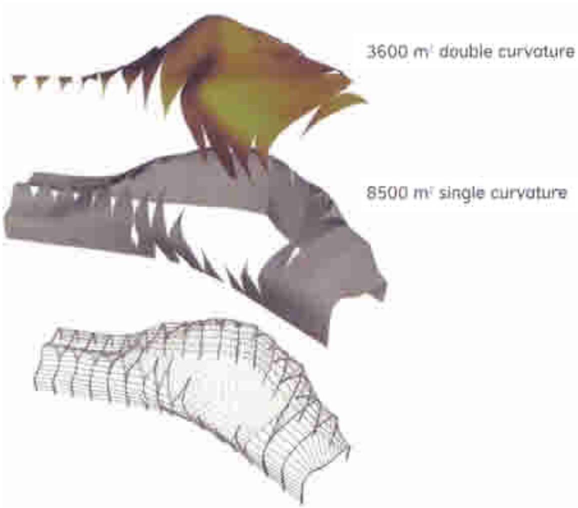


Fig. 14.22
An application of Gaussian curvature in architecture: Pompidou Two by NOX, invited competition for the City of Metz, France. The designed shape has been segmented into single curved areas and double curved parts. This segmentation is also reflected in the construction methodology.

Fig. 14.23
Minimal surfaces in form of soap films (images courtesy of K. Rittenschober). Minimal surfaces are characterized by vanishing mean curvature.

Visualization of the curvature behavior. The curvature behavior of a surface is difficult to convey by a shaded image. One means of visualizing such behavior employs color-coded images. The values of a given curvature measure—for example, Gaussian curvature K or mean curvature H (or another function of the principal curvatures)—are encoded according to color (Figure 14.24). In this way, minute imperfections of a surface can be made visible (see also Figure 14.16). The frequently used subdivision surfaces (Doo-Sabin, Catmull-Clark, Loop) possess a complicated and sometimes undesirable curvature behavior near irregular vertices of the base mesh; this can be visualized by curvature diagrams (Figure 14.25).

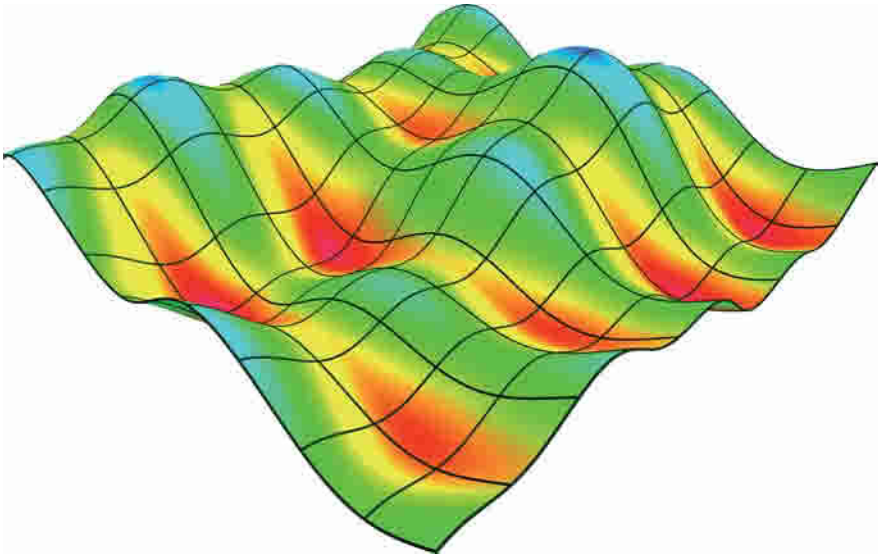


Fig. 14.24
Curvatures, here mean curvature, in a color-coded visualization are frequently used tools for surface analysis.

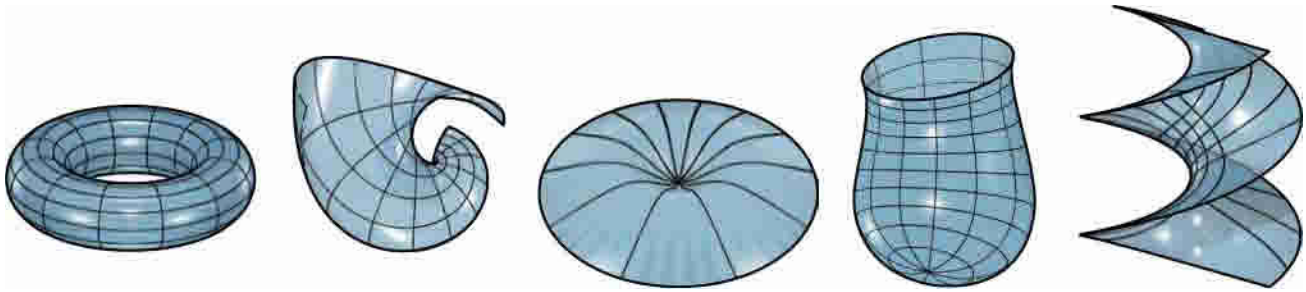
Fig. 14.25
The curvature behavior of a subdivision surface near an irregular vertex can be complicated and undesirable for certain applications.

This figure (courtesy of I. Ginkel and G. Umlauf) illustrates the problem for a Loop subdivision surface (left) via a color-coded Gaussian curvature diagram (right).

Principal curvature lines. To obtain an overview of the principal directions, one can use *principal curvature lines*. A principal curvature line is a curve on a surface whose tangents are in principal direction. Thus, through each general point of a surface there are two principal curvature lines that intersect at a right angle and touch the principal directions.

Some examples are shown in Figure 14.26. Principal curvature lines or related networks of curves are sometimes used for the generation of surface illustrations that aim at results that are similar to drawings made by artists. This is an instance of a *non-photorealistic rendering* technique (see Figure 14.27).

Fig. 14.26
The network of principal curvature lines of a surface represents fundamental shape characteristics.



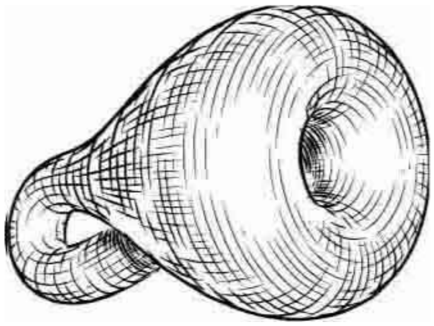
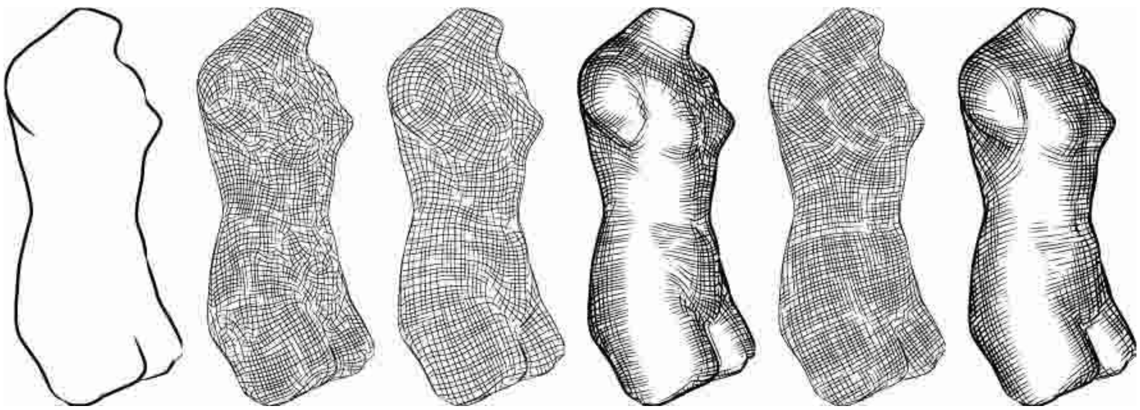
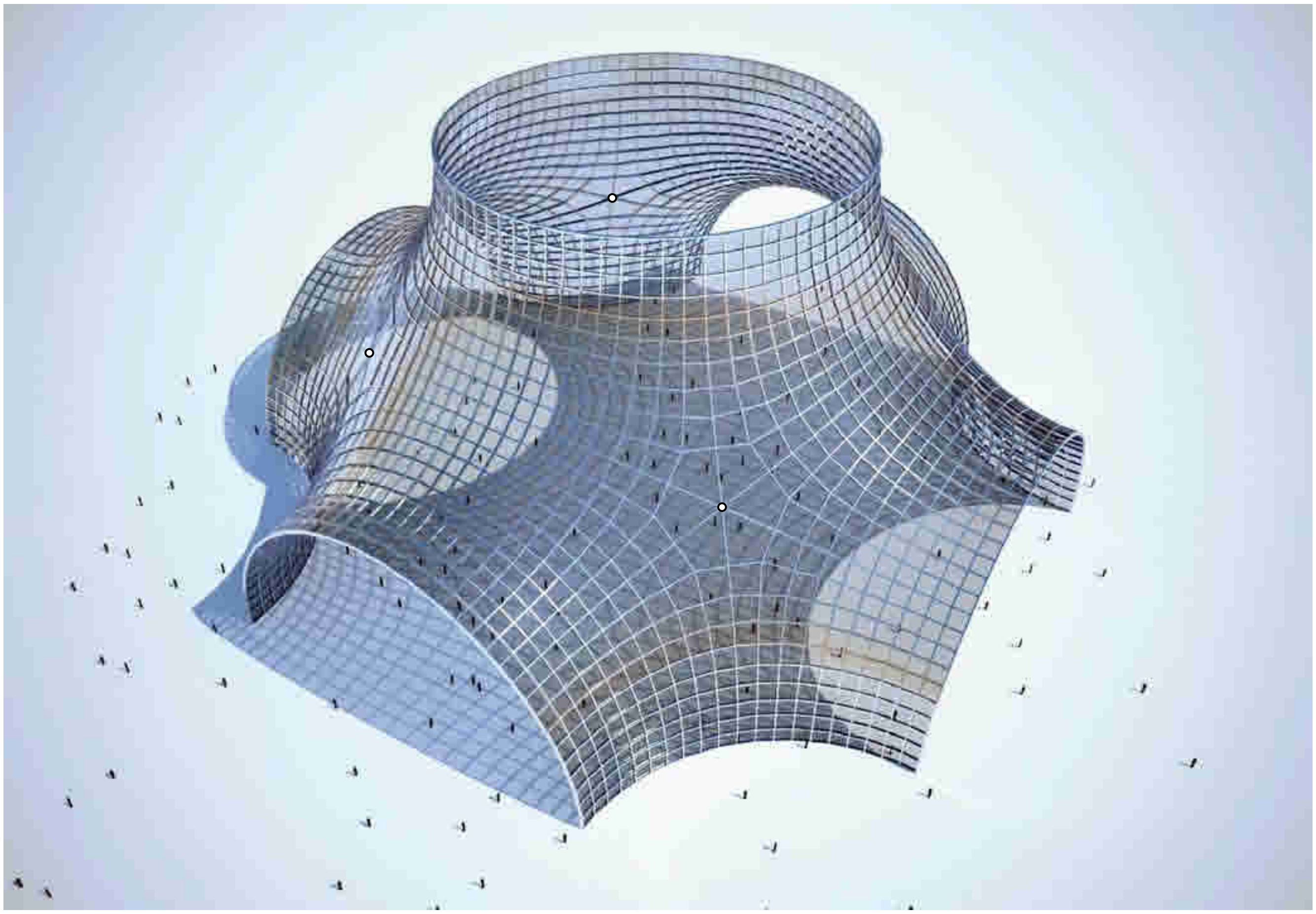


Fig. 14.27
Non-photorealistic rendering of a surface resembling a drawing. Hatching is guided by the principal directions on the surface. (Image courtesy of D. Zorin.)



Remark (umbilics). The principal directions are uniquely defined only if k_1 and k_2 are different. For $k_1 = k_2$, we have a special surface point called an *umbilic*. There, the osculating paraboloid P is a paraboloid of revolution or a plane ($k_1 = k_2 = 0$).
A sphere S (radius R) has only umbilics. The intersection curve with any plane through a surface normal is a great circle (radius R) and thus all normal curvatures equal $1/R$. At an umbilic, we have the same curvature behavior as for a sphere or a plane; in the latter case we also speak of a flat point (Figure 14.13). At an umbilic the network of principal curvature lines has a singularity (Figures 14.26 and 14.28).

Fig. 14.28
Principal curvature lines form the basis for the layout of freeform structures with planar quadrilateral glass panels. Here we only show such a structure. The geometric discussion for its generation is found in Chapter 19. The singular vertices in the mesh (valence 6) correspond to the umbilics (flat points) on an underlying smooth surface.



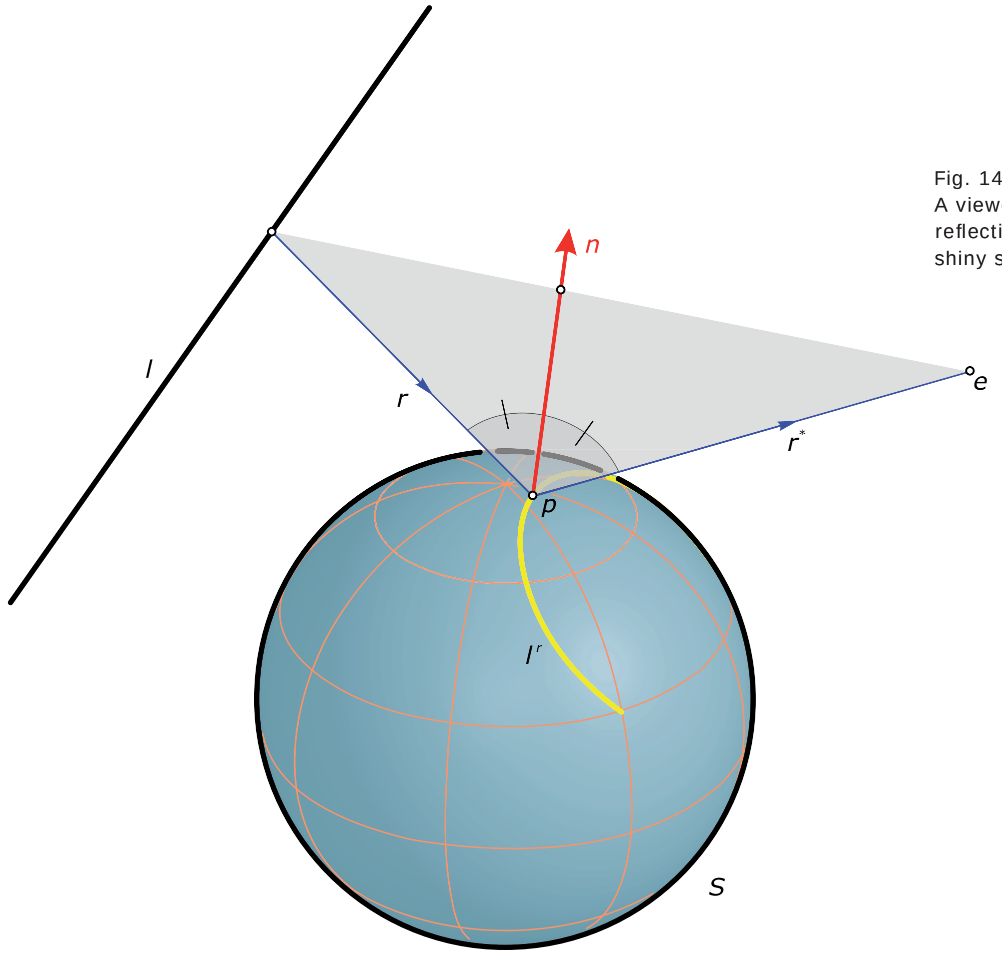


Fig. 14.29
A viewer with eye point *e* sees a reflection line l^r of a light source *l* on a shiny surface *S*.