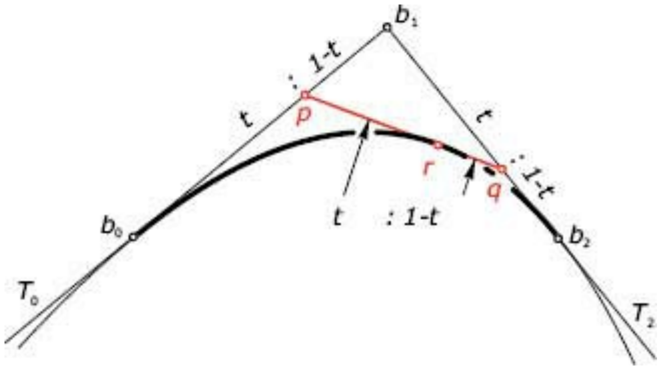

Fig. 7.31
The thread construction of a parabola uses repeated linear interpolation (see Geometry Primer) to generate points and tangents of the parabola defined by two contact elements (b_0, T_0) and (b_2, T_2).

Surfaces

We now generalize the concepts for curves to the study of surfaces. Whereas we have previously viewed curves as a one-dimensional series of points, we now consider surfaces as a type of two-dimensional skin in space. However, this rough representation of surfaces is too inaccurate for studying them in detail. Thus, analogous to the curves we introduce parametric, explicit, and implicit representations of surfaces for mathematically handling surfaces and studying their geometry analytically.

Parametric representation. In contrast with curves, the coordinates of a surface point depend on two different parameters u and v. Thus, a *parametric surface* S can be represented by $\mathbf{p}(u,v) = (x(u,v), y(u,v), z(u,v))$, where the parameters u and v assume all values in a two-dimensional region R (Figure 7.32). Instead of mapping a one-dimensional interval I into space (curve case), we now have a continuous mapping of a two-dimensional region R into space.

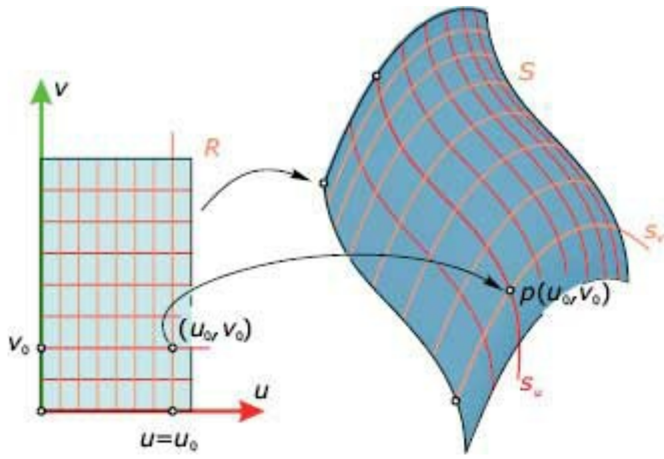

Fig. 7.32
The parametric representation describes a mapping from a region R of the (u,v)-parameter plane to a surface patch S in three-dimensional space.

Every pair of parameters u and v that defines a point (u,v) in the region R is mapped to a surface point $p(u,v)$. Analogously with curves, we call the functions $x(u,v)$, $y(u,v)$, and $z(u,v)$ *coordinate functions* and $\mathbf{p}(u,v)$ a *parameterization* of S. When we fix the parameter $u = u_0$, we obtain a *parameter curve* or *v-line* s_v on the surface. The name v-line expresses that v varies (i.e., v is the curve parameter).

On the other hand, a *u-parameter curve* or *u-line* s_u is generated on the surface S when we fix the parameter v. The sketching of parameter lines is often useful in visualizing the spatial structure of a surface. Thus, they can be used as an architectural tool. Figure 7.33 illustrates the benefits of parameter lines.

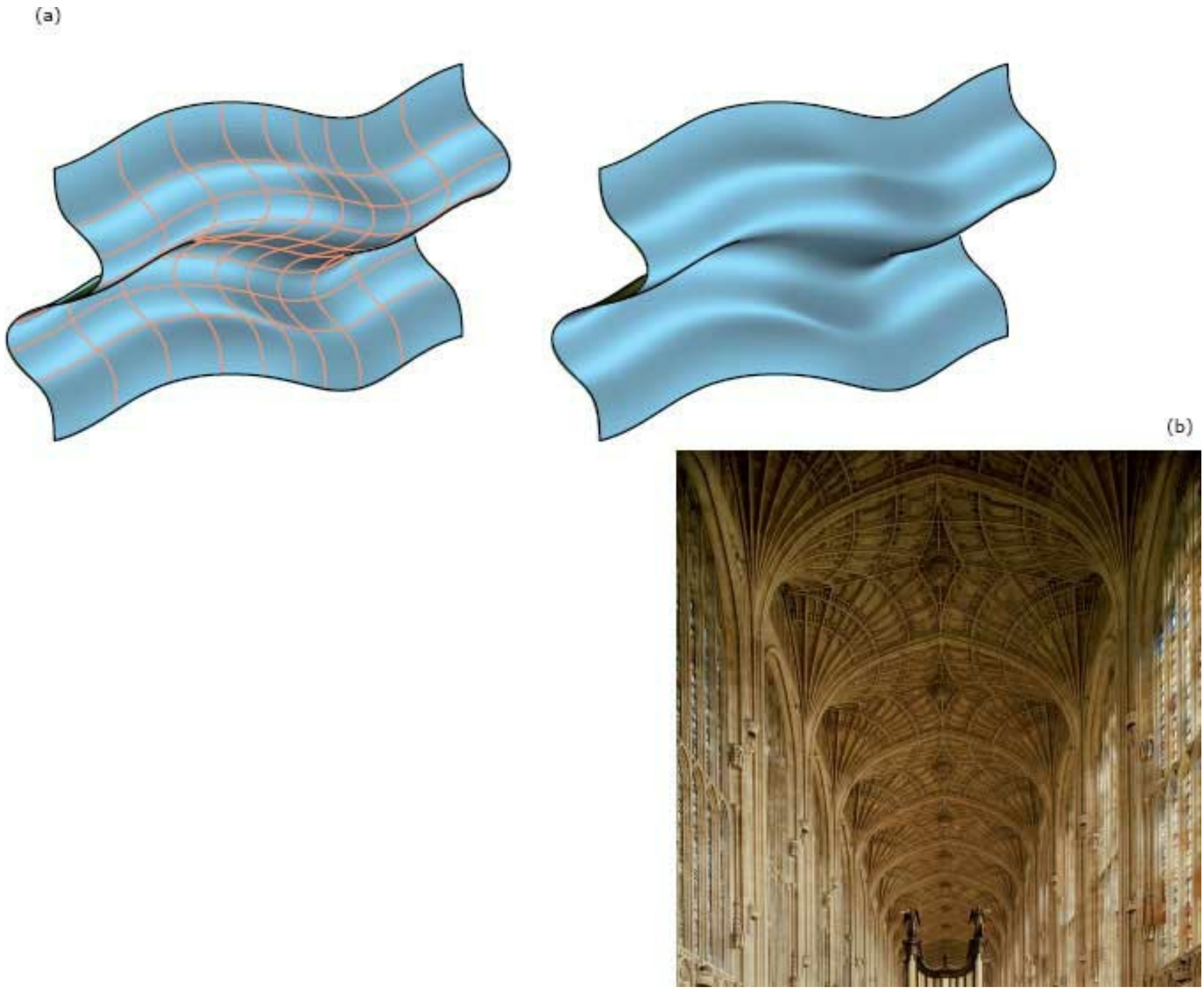

Fig. 7.33
Parameter lines support the spatial impression of surfaces.
(a) The same surface with and without parameter lines.
(b) The usage of parameter lines in architecture is illustrated by means of the fan vault of the *King's College Chapel* (1446–1515) in Cambridge.

Example:

Parametric representation of a sphere. Given are the center $m(0,0,0)$ and the radius r of a sphere. According to Figure 7.34, we derive the coordinates of a sphere point p as $\mathbf{p}(u,v) = (r \cdot \cos(u) \cdot \cos(v),\ r \cdot \cos(u) \cdot \sin(v),\ r \cdot \sin(u))$.

If the parameters u and v assume all values in $[-\pi/2, \pi/2]$ and $(-\pi, \pi]$ respectively, we obtain the entire sphere. Then the v-parameter curves are circles (latitude u = const) in planes parallel to the xy-plane. The u-parameter curves are meridian circles (longitude v = const) running through the north and south poles.

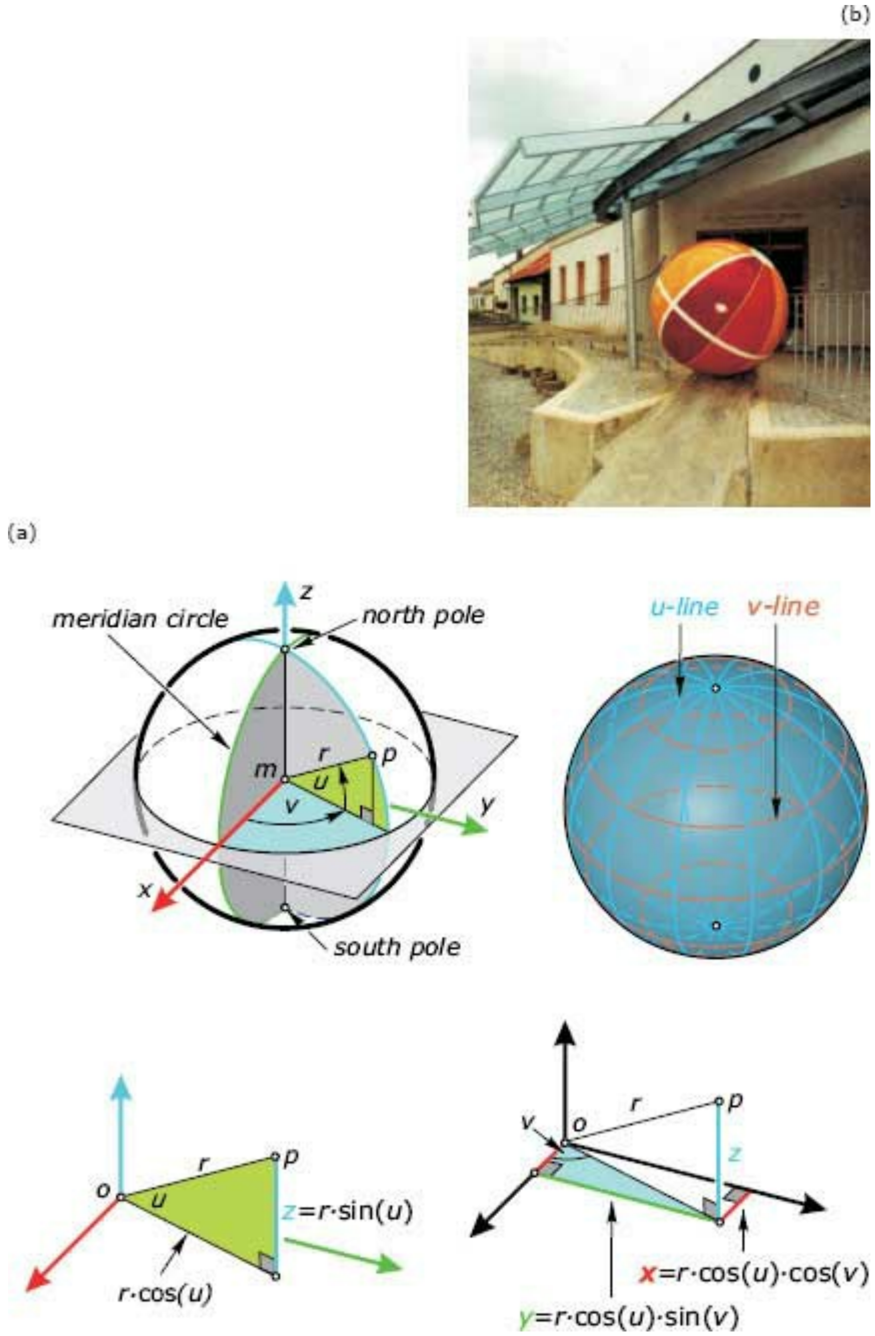

Fig. 7.34
(a) A parametric representation of a sphere can be based on spherical coordinates; namely, the geographic latitude *u* and longitude *v*. The meridian circles run through the north and south poles of the sphere.
(b) The *Rolling Ball* (1992) in Seyring by Richard Künz (image courtesy of E. Mrazek).

Example:

Cylinder. A surface with a parameterization of the form $\mathbf{p}(u,v) = (x(u), y(u), v)$ is a general cylinder, where $c = \mathbf{c}(u) = (x(u), y(u), 0)$ is its base curve in the *xy*-plane. All *u*-lines are congruent to *c* and lie in planes parallel to the *xy*-plane.

The rulings of the cylinder are the *v*-lines. They are parallel to the *z*-axis. Figure 7.35 shows two examples, with $\mathbf{c}(u) = (2\cdot\sin(u), 3\cdot\cos(u))$—respectively $\mathbf{d}(u) = (2\cdot\cos(u) + 2\cdot\cos(2\cdot u), 2\cdot\sin(u) + 2\cdot\sin(2\cdot u))$—as *u*-curves in the *xy*-plane.

Example:

Tangent surface. If $\mathbf{c}(t)$ denotes a parametric representation of a spatial curve *c*, we have shown that the curve tangent at point $\mathbf{c}(t)$ is described by $\mathbf{c}(t) + u.\mathbf{c}'(t)$. If we consider *t* varying, we obtain the set of all tangents of *c*. This so-called tangent surface (Figure 7.36) has the parametric representation $\mathbf{p}(u,t) = \mathbf{c}(t) + u.\mathbf{c}'(t)$. The *u*-lines ($t$ = const) are the tangents. Tangent surfaces have important geometric properties (discussed in Chapter 15, on developable surfaces).

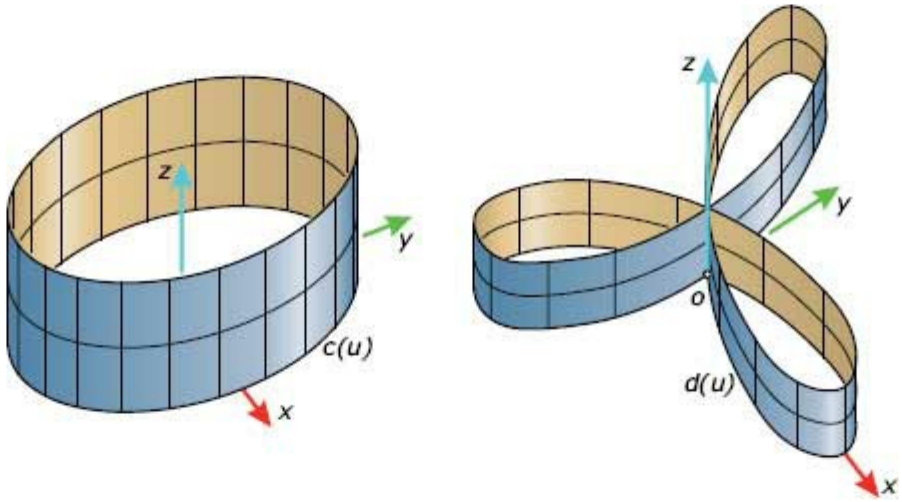

Fig. 7.35
Cylinders with planar u-lines.

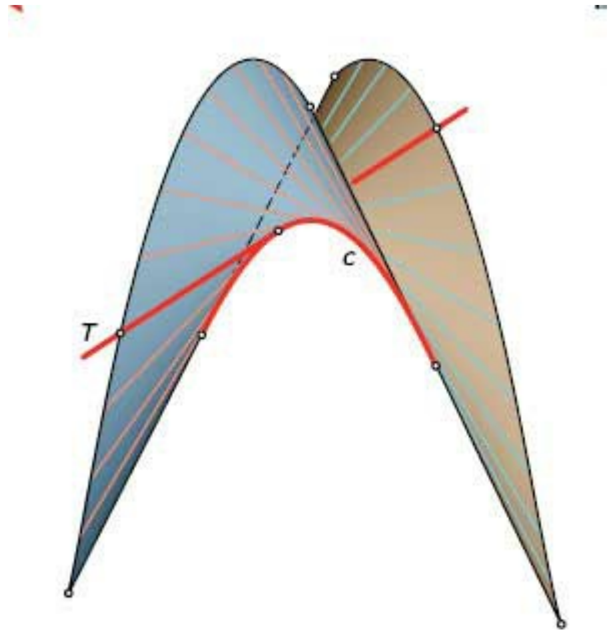

Fig. 7.36
A surface formed by the tangents of a spatial curve.

Explicit and implicit representation. A surface S can also be seen as a set of all points that satisfy a condition of the form $z = f(x,y)$ or $F(x,y,z) = 0$. We call the surface description $z = f(x,y)$ an *explicit representation* and $F(x,y,z) = 0$ an *implicit representation*. The explicit representation $z = f(x,y)$ is mainly used for visualization of a function $f(x,y)$ of two variables. We also call this surface the *graph* of the function $f(x,y)$. Of course, the explicit representation is a special case of an implicit representation.

As an example, we illustrate in Figure 7.37 a hyperbolic paraboloid (see Chapter 9) with the explicit representation $z(x,y) = 2x^2 - 3y^2$ and the "chair" surface with the implicit representation $(x^2 + y^2 + z^2 - ak^2)^2 - b[(z - k)^2 - 2x^2].[(z + k)^2 - 2y^2] = 0$, where $k = 5$, $a = 0.95$, and $b = 0.8$.

Tangent plane and surface normal. When we replace the parameters u and v in the parametric representation of a surface respectively with arbitrary functions $u(t)$ and $v(t)$, we obtain a curve $\mathbf{c}(t) = (x(t), y(t), z(t))$ on the surface S. Curves on S are also called *surface curves*. We could also say that the surface parameterization maps the curve $(u(t),v(t))$ in the parameter plane to a surface curve. The tangent t_c of such a curve in a point p is called a *surface tangent*.

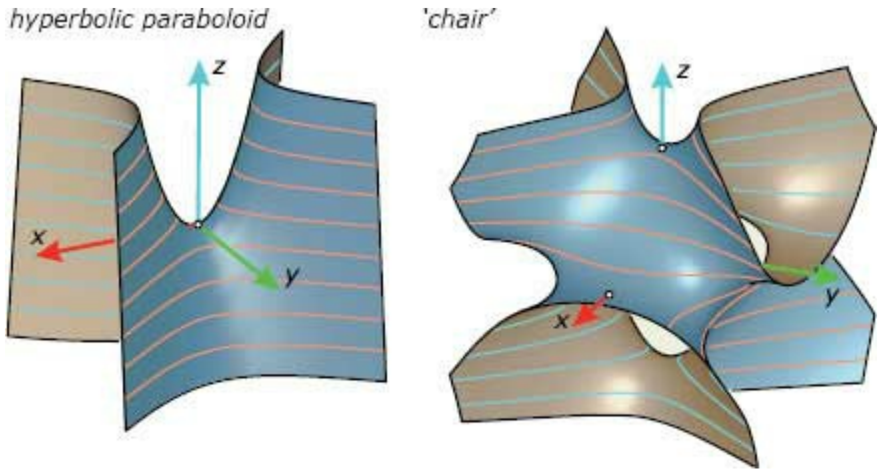

Fig. 7.37
A hyperbolic paraboloid and a "chair" surface serve respectively as examples of surfaces in explicit and implicit representation.

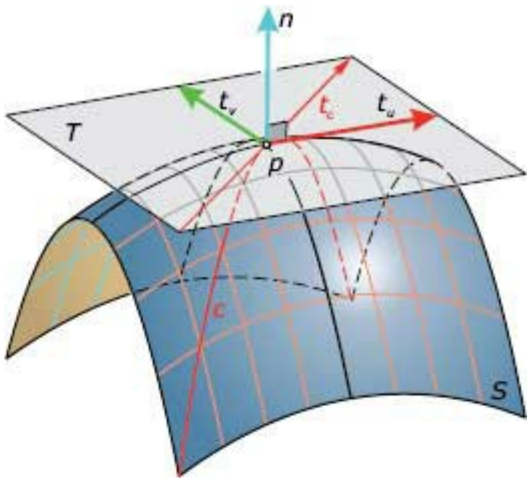

Fig. 7.38
In a regular point p, the tangents of the parameter lines span the tangent plane. Any surface curve passing through p has a tangent in p, which lies in the tangent plane.

In a regular point of the surface, all surface tangents lie in a plane—the *tangent plane* of the surface S in the point p. In this case, the tangents t_u and t_v of the parameter lines define the tangent plane (Figure 7.38). For the reader familiar with basic analysis, we note that the partial derivative vectors of $\mathbf{p}(u,v)$ with respect to u and v are respectively the directions vectors of t_u and t_v. The *surface normal* n is the straight line through p orthogonal to the tangent plane.

Points of a surface, like the apex of a cone, where no unique tangent plane exists are called *singular points*.

Example:

Whitney umbrella. The surface with the parameterization $\mathbf{p}(u,v) = (u, v^2, uv)$ is illustrated in Figure 7.39. Along the y-axis, this surface has a line of self-intersection. At any point on the positive y-axis, we have two different tangent planes. Thus, all points belonging to the surface curve $\mathbf{c}(v) = (0,v^2,0)$ with constant parameter $u = 0$ are singular points. All other points turn out to be regular ones.

Example:

Surface with many singularities. Figure 7.40 shows a remarkable surface with an implicit representation $F(x,y,z) = 0$, where F is a certain polynomial of degree 7. This surface, known as *Labs septic*, carries 99 singular points.

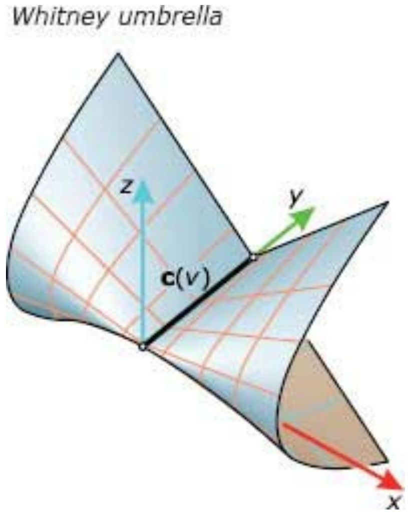

Fig. 7.39
Whitney’s umbrella carries a line with singular points.

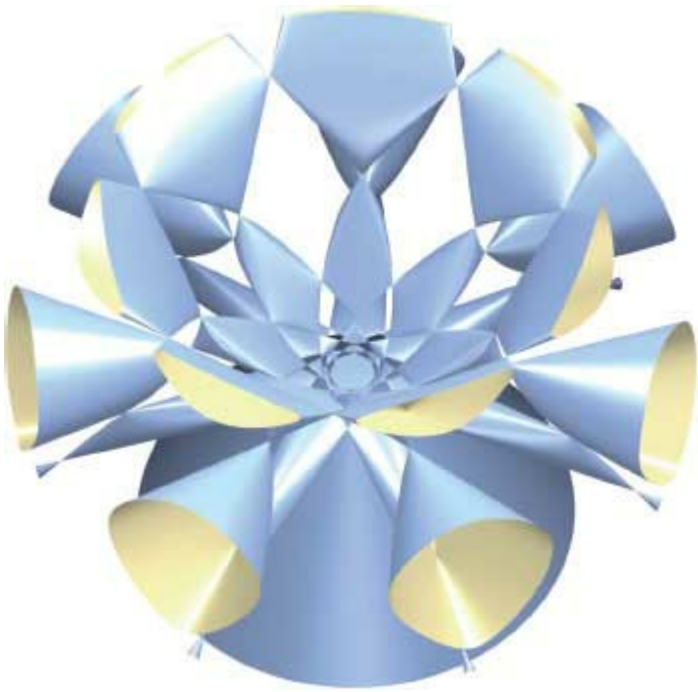

Fig. 7.40
View of the inner part of the *Labs septic* (image courtesy of Oliver Labs).

Note that even in the proximity of a surface point the tangent plane can contain more surface points than the touching point p. Figure 7.41 illustrates three different types of the behavior of a tangent plane at a point. According to this, we distinguish among *elliptic*, *hyperbolic*, and *parabolic surface points*. We will study these point types in detail in Chapter 14 (on visualization and analysis of shapes).

Contour and apparent contour. When we sketch a surface S or a CAD program produces an image of it, we need the *contour* of the surface to distinguish between visible and occluded parts of the surface. To obtain the contour, we first define the *contour generator* c^g as a set of all points p on S whose tangent plane T contains the projection ray through p (Figure 7.42).

In the case of a central projection, this implies that T passes through the eye point e. In the case of a parallel projection, T is parallel to the projection rays. In an equivalent formulation, we can say that all projection rays tangent to S form a cone (in the case of a central projection) or a cylinder (in the case of a parallel projection). The contour curve between cone (or cylinder) and S is the contour generator.

The projection of the contour generator c^g is the *apparent contour* c^a. The projection of a point p of c^g yields a point p' of the apparent contour c^a. Because p is on c^g, the image of its tangent plane T is a straight line T'. This line T' is the tangent of the apparent contour c^a in p' (Figure 7.42).

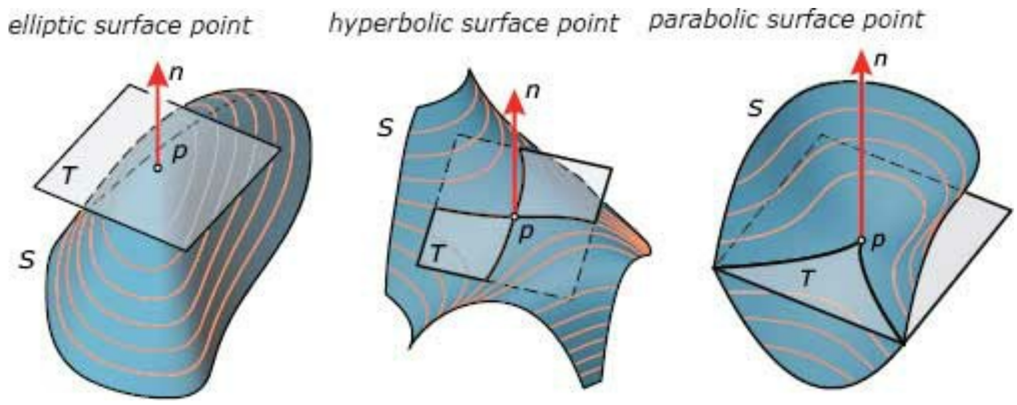

Fig. 7.41
Elliptic, hyperbolic, and parabolic points and the behavior of the tangent plane in these points.

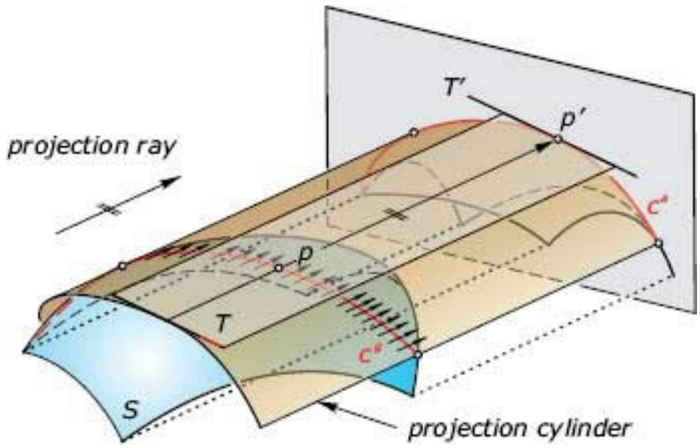

Fig. 7.42
The apparent contour is the projection of the contour generator. The latter is the set of all surface points p where a projection ray touches S.

Figure 7.43 illustrates an interesting property in connection with contours and surface curves. If the curve tangent t_c of a surface curve c is not a projection ray, the image t_c' touches the apparent contour in the point p' (Figure 7.43a). Otherwise, if the tangent t_d of a surface curve d is a projection ray the image d' of the curve d possesses a cusp in the point p' (Figure 7.43b). Because contour generators are already defined via projection rays tangent to the surface, the special event that a projection ray touches the contour generator is not as unlikely. Hence, we quite often find cusps in apparent contours. There, the visibility of the contour may switch (Figure 7.43c).

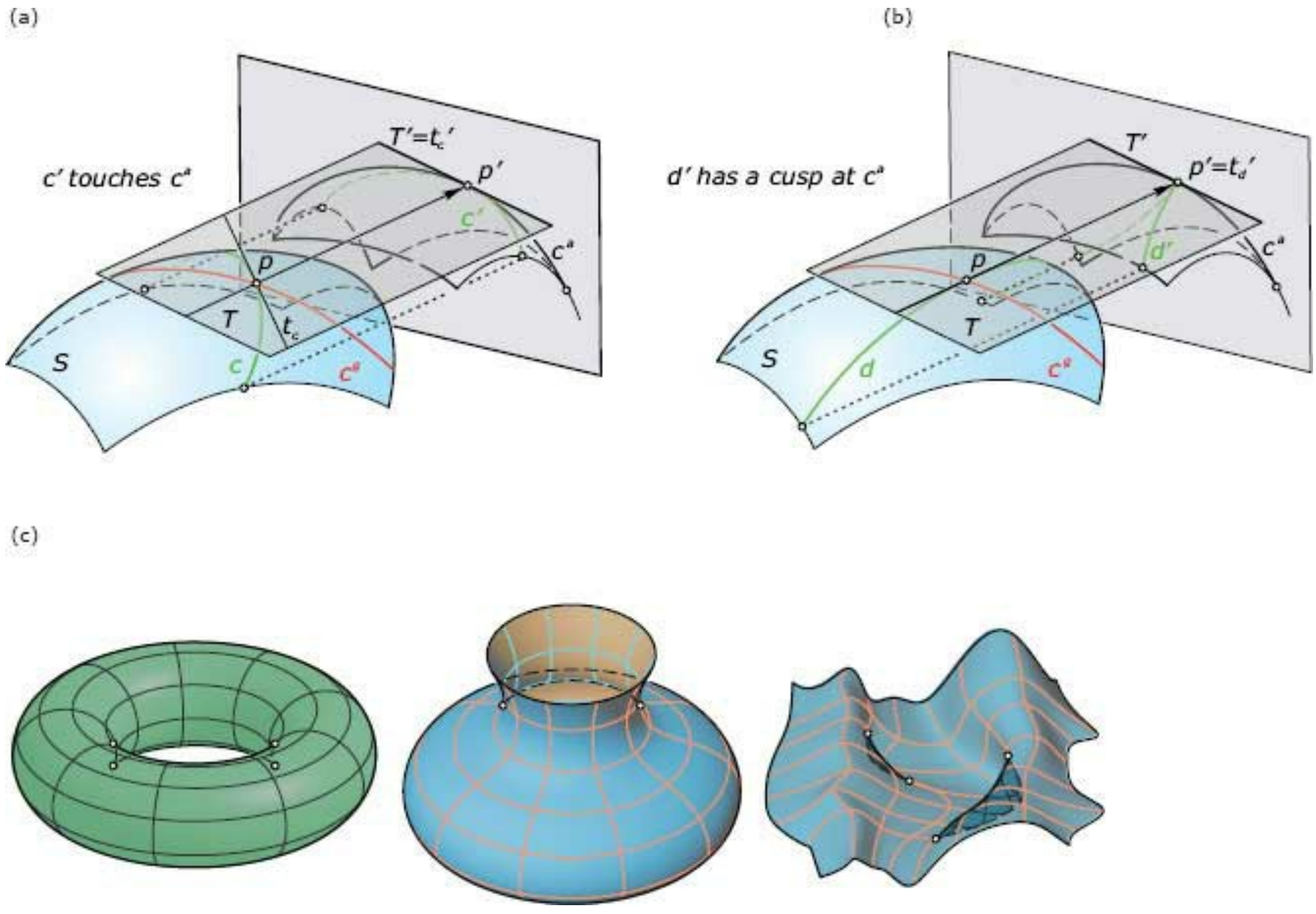

Fig. 7.43
The behavior of the projections of surface curves.
(a) In general, the projected curve c' touches the apparent contour c^a.
(b) If a projection ray is tangent to a curve *d*, the projected curve d' exhibits a cusp.
(c) Cusps in apparent contours arise from projection rays which are tangent to the contour generator.

Intersection Curves of Surfaces

In Chapter 4 (on trimming and splitting) we studied intersection curves of two surfaces. We used appropriate CAD tools to generate the intersection curves without any understanding of the process of how to find these curves. With a better understanding of curves and surfaces, in the following we discuss some geometric background of intersection curves. This material shows why a stable implementation of surface/surface intersection is a challenge in the development of any CAD program, and it makes us less critical if this tool sometimes fails. Moreover, it provides some hints for correctly sketching intersection curves by hand.

Constructing points via auxiliary planes. To find points of the intersection curve(s) of two surfaces, we may use a set of auxiliary planes. These auxiliary planes intersect the original surfaces along surface curves c_1 and c_2. The common points of these planar curves are points of the intersection curve c. If we find appropriate auxiliary planes that intersect both surfaces along simple curves c_1 and c_2, we are able to construct points of the intersection curve. Only for very special surfaces can we actually find such auxiliary planes.

Figure 7.44a illustrates this method for surface/surface intersection in the case of two cylinders. In this case, planes parallel to the generators of both cylinders cut the cylinders along generators. Thus, points of the intersection curve can be found by intersecting straight lines.

The same method can be applied to the construction of the intersection curve of a sphere and a cone. As shown in Figure 7.44b, points of the intersection curve can be found as common points of a pair of straight lines and a circle. We use auxiliary planes through the apex of the cone. In general, these planes intersect the cone along a pair of rulings and the sphere along a circle. As we have seen in both examples, the method of auxiliary planes is an appropriate one when cylinders, cones, or spheres are involved.

Use of auxiliary spheres. Points of the intersection curve of a rotational cylinder and a rotational cone with intersecting axes can also be found with the help of auxiliary spheres. Spheres with center *m* in the intersection point of the two axes *a* and *b* intersect the cylinder and the cone along circles. Their common points belong to the intersection curve *c* (Figure 7.45).

This method is best used with rotational surfaces (see Chapter 9) with intersecting axes. In Figure 7.46b, we use a projection orthogonal to the plane spanned by the two rotational axes. Then, the images c_1' and c_2' of the circles c_1 and c_2 are straight line segments and the points of the intersection curve c' can be found as common points of c_1' and c_2'.

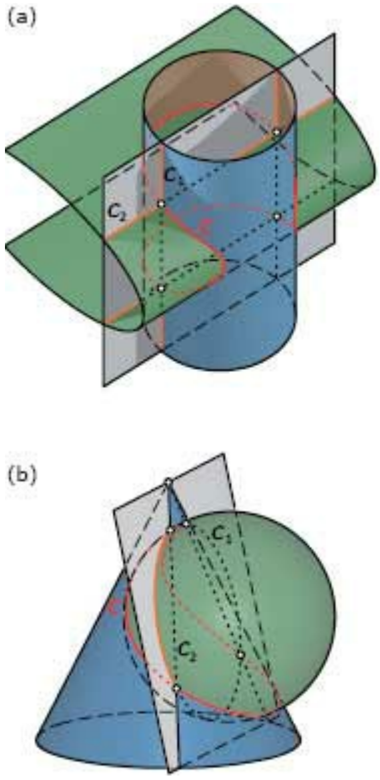

Fig. 7.44
Points of an intersection curve may be constructed with the help of appropriate auxiliary planes.

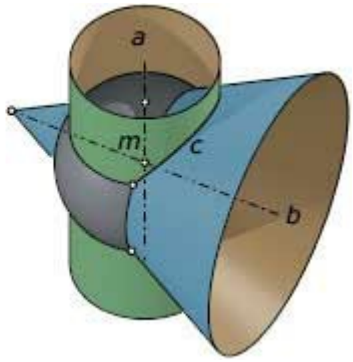

Fig. 7.45
Construction of points of an intersection curve using auxiliary spheres.

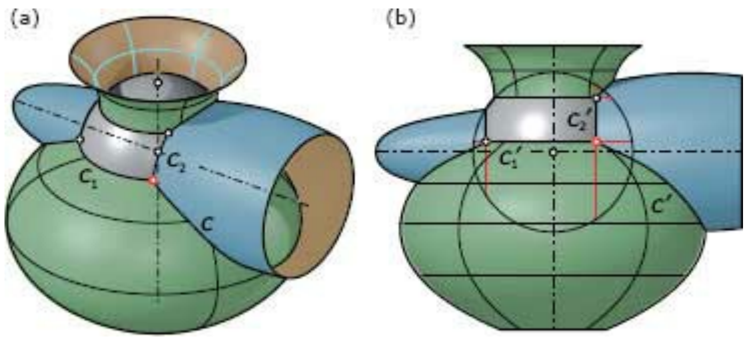

Fig. 7.46
The use of a special projection orthogonal to both axes simplifies the pointwise construction of the intersection curve of two rotational surfaces.

Tangents of intersection curves. When sketching an intersection curve by hand, it is often better to construct a few points plus their tangents instead of generating many points without tangents. The tangents of a surface curve are contained in the respective tangent planes (Figure 7.38). Thus, the tangent t_p in a point p of the intersection curve is the intersection line of the tangent planes T_1 and T_2 of the two surfaces in the point p (Figures 7.47 and 7.48).

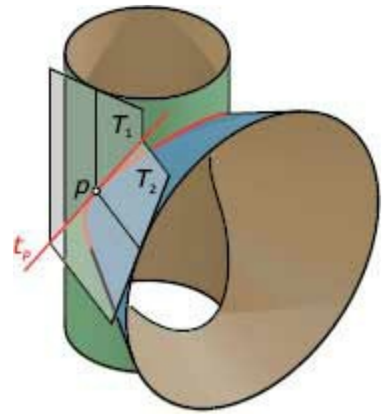

Fig. 7.47
The tangent of the intersection curve is contained in the tangent planes T_1 and T_2 of both surfaces.

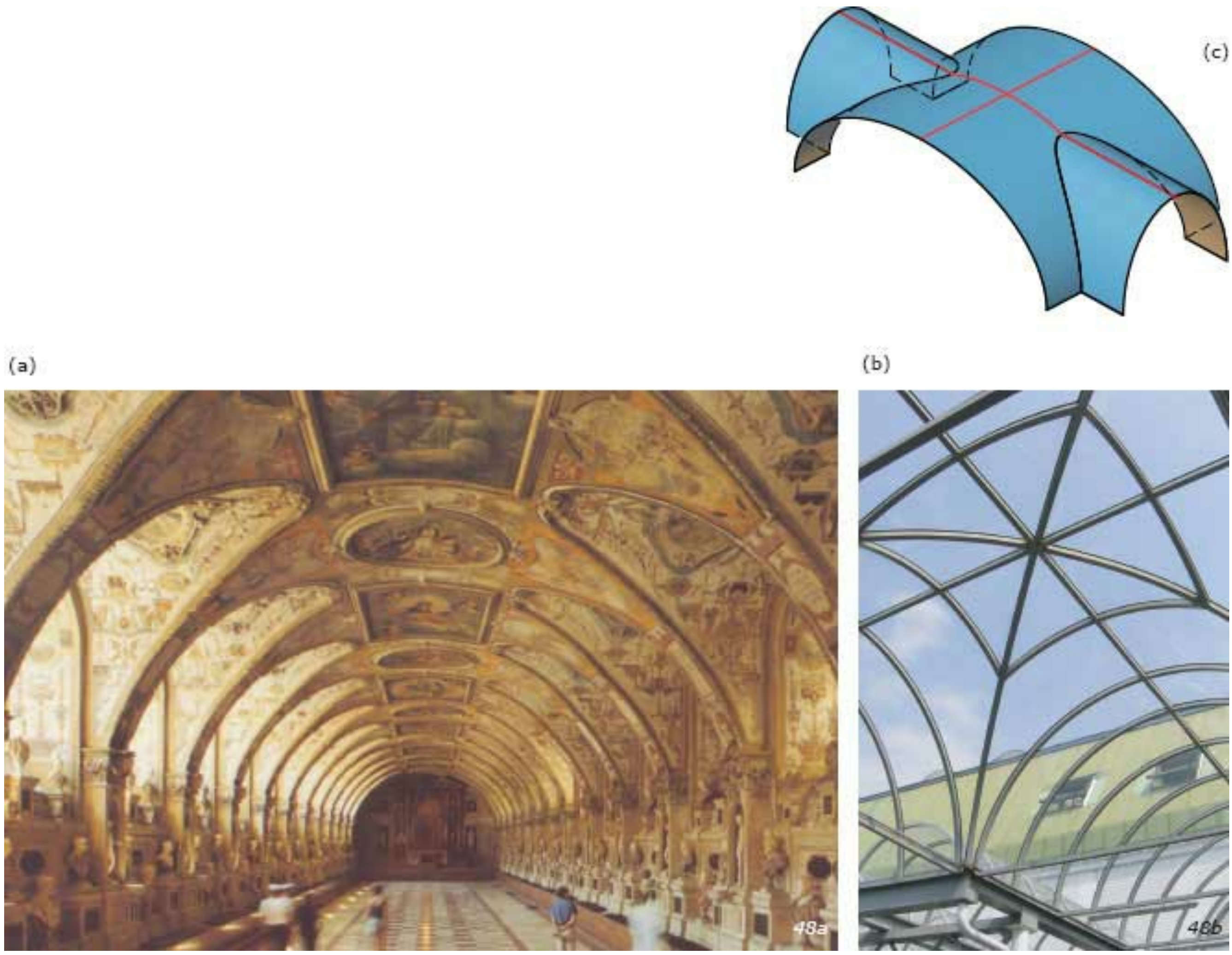

Fig. 7.48
In architecture, vaults often feature interesting intersection curves (including parts of conic sections).
(a) Intersection curves in the *barrel vault of the Residenz* (1569–1571) in Munich.
(b) A cross vault in a building entrance (image courtesy of Martin Reis).
(c) Model of a vault.

This construction fails if the given surfaces have the same tangent plane at p. Such points of tangency usually lead to double points of the intersection curve. Figure 7.49 shows an example: the common points of the two ellipses along which the two cylinders intersect are exactly their common points of tangency.

Conic sections as intersection curves. Conic sections as special intersection curves can be found in many applications in the building and constructing industry (Figure 7.48). This principle was extensively used to construct vaults. It can be generalized to the following statement.

If two rotational cylinders or cones possess a common inscribed sphere, their intersection curve decomposes into a pair of conic sections or into a conic section and a single ruling.

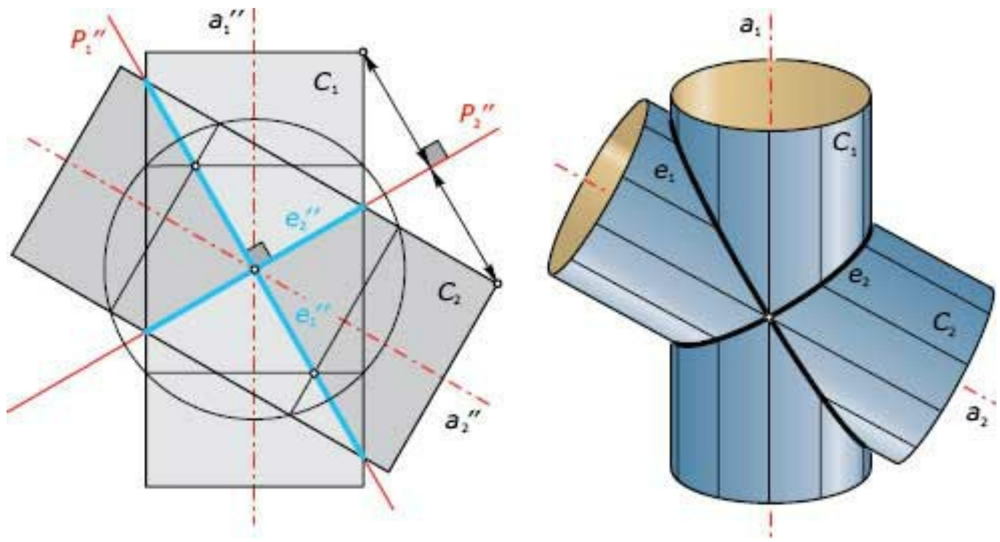

Fig. 7.49
The complete intersection curve of two rotational cylinders with intersecting axes and the same radius is a pair of ellipses in orthogonal planes.

Example:

Rotational cylinders with intersecting axes and equal radius. Given are two rotational cylinders C_1 and C_2 with intersecting axes and equal radius r. We assume the cylinder axes a_1 and a_2 to lie in the *yz*-plane (Figure 7.49). Using auxiliary spheres for the generation of points of the intersection curve, we recognize that the front view of the intersection curve consists of two straight line segments e_1'' and e_2'' . These line segments are parts of the angle bisectors of a_1'' and a_2''. Thus, the intersection curve consists of two planar curves e_1 and e_2.

Due to the fact that planar intersection curves of rotational cylinders are ellipses, the complete intersection curve consists of two ellipses e_1 and e_2 in orthogonal planes P_1 and P_2. These planes are the bisecting planes of the axes a_1 and a_2. Reflecting one cylinder at such a bisecting plane, we obtain the other cylinder. Hence, the intersection curve between a bisecting plane and a cylinder also belongs to the other cylinder and thus to the intersection curve. This argument may also help in understanding intersection curves in other symmetric surface/surface configurations.

Figure 7.50 illustrates planar intersection curves of two rotational cones. We will study an even more general application of this statement in Chapter 9 (on intersection curves of quadrics).

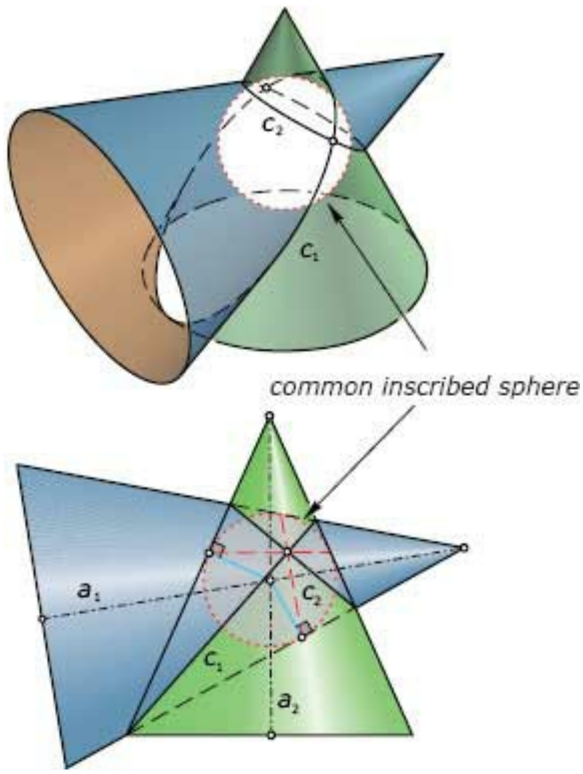

Fig. 7.50
The intersection curve of two rotational cones with a common inscribed sphere decomposes into conic sections.

Space curves as intersection curves of general cylinders. At the beginning of this chapter, we studied the parametric representation of spatial curves. For planar curves, we also introduced explicit and implicit representation. What about implicit representations of spatial curves?

As an example for an explicit representation of a planar curve, we used the parabola c: $y = x^2$—which we derived from the parametric representation $\mathbf{c}(t) = (t, t^2)$. In a similar way, the spatial polynomial cubic curve $\mathbf{d}(t) = (t, t^2, t^3)$ satisfies *two* independent equations $y = x^2$ and $z = x^3$.

These two equations define two cylinders C_1, C_2 with generators parallel to the z-axis and y-axis, respectively. Their base curves in the coordinate planes $z = 0$ and $y = 0$ are the parabola $y = x^2$ and the cubic $z = x^3$, respectively. Thus, the spatial polynomial cubic can be generated as an intersection curve of the two cylinders (Figure 7.51). This example shows that a space curve is defined by at least two implicit equations. However, due to additionally occurring intersection curves we may need even more equations to uniquely define the curve.

This is seen with the same example if we replace the equation $z = x^3$ with the equation $y3 = z^2$. It is also satisfied by = (t, t^2, t^3) and describes a cylinder C_3 with rulings parallel to the x-axis and a cubic base curve (with a cusp; Figure 7.52) in the yz-plane. Our spatial polynomial cubic is also an intersection curve of the parabolic cylinder C_1 and the cubic cylinder C_3. However, the complete intersection of these two cylinders consists of a second spatial cubic d—which arises from the first spatial cubic by reflection at the yz-plane.

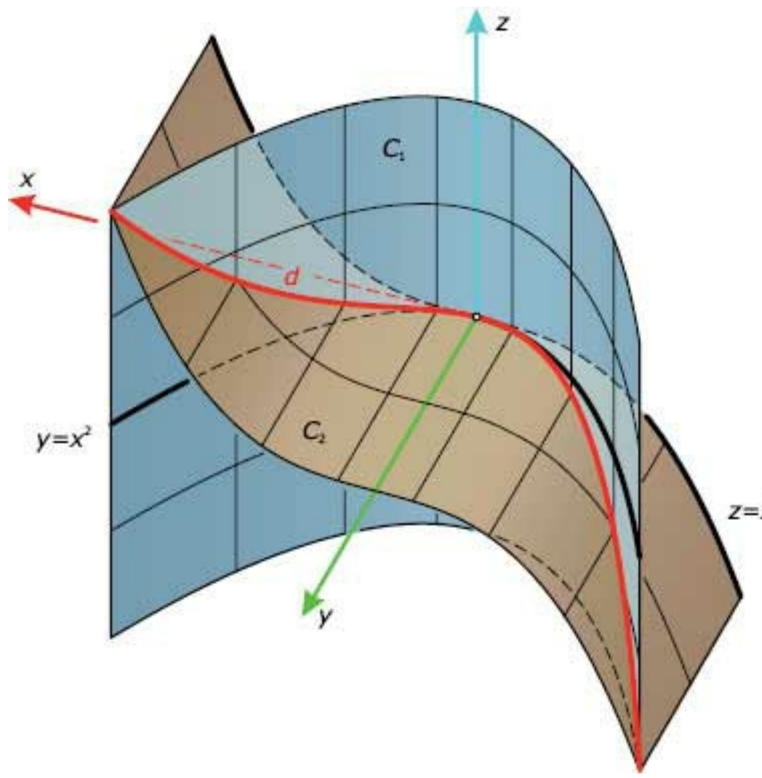

Fig. 7.51
A parametric cubic curve as an intersection curve of two cylinders.

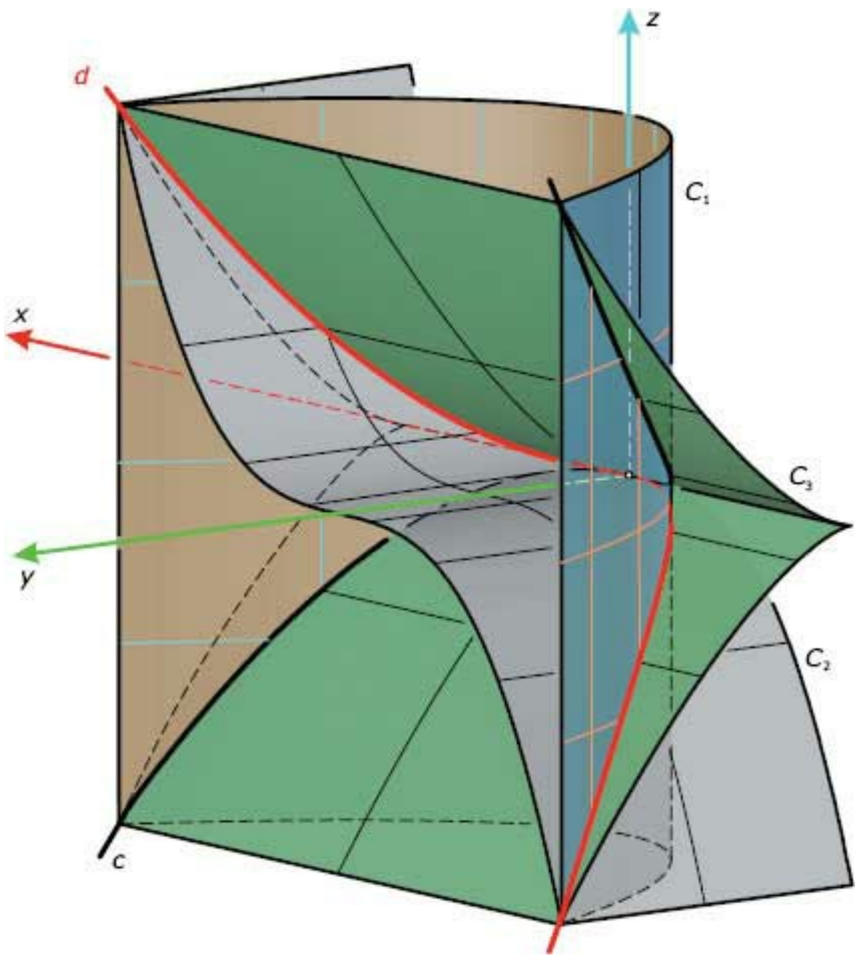

Fig. 7.52
The parametric cubic curve of Figure 7.51 as an intersection curve of a parabolic cylinder and another cubic cylinder. Here, the complete intersection contains a further spatial cubic.