Imperial Porphiry and Golden Leaf: Sierpinski Triangle in a Medieval Roman Cloister[1]

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Abstract. In medieval churches motives are found, similar to what we call today "Sierpinski triangle": a same composition of full and void areas, interweaved and repeated at smaller and smaller scale. The motive has seen its mathematically rigorous definition in 1915, and has been a "benchmark" for scientists thereafter. Mathematicians imagine and study what would remain upon carrying on indefinitely the procedure of inserting voids: a "powder of points" would be left, organized in a precise way around large voids. On the other hand, the geometrical compositions of the Marmorari Romani, loosely known as "Cosmati", are often characterized by repetitions on different spatial scales, thus suggesting to the spectator an in-depth view [22, 30]. Actual artifacts composed iteratively can be reliably analyzed with mathematical methods, if the motive shows at least three levels of iteration (i.e. different spatial scales). In [10] we reviewed some such triangles, in stone, isolated in medieval floors.

In the present paper we report about some new examples recently found in Rome, and not published yet in any form. These are isolated Sierpinski triangles in golden leaf, contained in the frieze of medieval cloisters. Their composition is iterated at 3 and 4 levels. Moreover their placement warrants to their authenticity. Where the stones have fallen out, empty lodgings along the frieze testify that it has gone untouched for a long time. The "protagonists" of the motive are the smallest stones, as they are the ones in golden leaf, i.e. all that is perceived when looking up. Moreover the golden leaf reflects the light, adding the perceptually smaller spatial scales of the shimmering.

Keywords: Marmorari Romani, Sierpinski triangle, Cosmatesque art, visual perception

1 Introduction: Our Mental Eyes, Bridging Across Times

Mathematics and physics research, starting from the mid-nineteenth century, have tuned our collective mental eyes to study phenomena structured on many different spatial scales, down to the infinitesimal, and tools were developed to classify and measure them. New concepts, words and structures were forged for studying critical phenomena in Physics (Statistical Mechanics) and testing the Continuum hypothesis and the concept of dimension in Mathematics (Topology). One such tool is to define recursively sets displaying the same organization at smaller and smaller scale, and then to use them as models, allowing and testing the description of natural phenomena that so far had eluded description (quantitative and qualitative). What they all had and have in common, loosely speaking, is an intrinsically discontinuous geometry. Starting with the classic "ternary Cantor set" in 1883 [5], mathematicians defined over the decades more and more sets recursively, (i.e. by an iterative process) subdividing a given set indefinitely; these mathematical objects provided rigorous examples for very abstract, speculative and founding problems as the Hypothesis of continuum, and the very concepts of dimension and measure. More recently, they are providing the theoretical framework for studying chaotic dynamics, and its strange attractors, often displaying such interwoven structures.

Sierpinski triangle, or gasket [27], dating form 1915, is one of these seminal sets and still to this day provides a benchmark to test new ideas in different scientific realms [26, 29]. This entire general frame of scientific thought has also culturally tuned our eyes to detect similar organizations in actual art and architecture, i.e. in man-made artifacts. Recursive subdividing procedures have been observed in the designs of the mosaic medieval floors of central Italy, in the works internationally known as Cosmati floors, due to a group of families today referred to as *Marmorari Romani* in scholarly literature. Motives of the cosmatesque repertoire, including these triangles, can be found in the classic book by Jones [17]. These motives keep being used [12, 13]. In this paper, as in the previous, we only deal with medieval ones.

The idea of a self-similar organization in Cosmati pavements was already suggested in [22, 30], together with a perceptive interpretation, namely that the glance is invited to the smaller scale, as in a zooming, thus providing a sense of depth. In a previous paper [10] we went further in this direction, introducing the idea of a "self-similar carpet", and

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documenting explicit appearance of a Sierpinski isolated triangle on the floor of several Romanesque churches in Rome and in central Italy. In all studies and documentations of man-made artifacts one has to be clear as to the assumptions. The standard mathematical tool is the use of some concepts developed for "fractals". Obviously, though, no man-made artifact organization can be recursively iterated on infinitely many spatial scales, down to the infinitesimal. In this regard, we choose to abide by the recommendation made by the Yale group [31]: in order to make the case for a fractal in an artifact, this should show at least three clear scale levels of iterations. We add that sufficient iterations assure two aspects: one cultural, and one perceptive. First, is the understanding and will, on the part of the artisan, to accomplish an iteration procedure on several different scales. On the perception side, three levels ensure that while looking at the set, one of the essential features of fractal is suggested, i.e. that the limit set of the iteration process be a Cantor set (a powder of points), and not isolated points, as would happen in other more familiar rescaling procedures such as rescaling around a central point. Our mind naturally does such actions, as Federigo Enriquez said "…how could such experiences have a cogent value beyond the limits in which they had actually been experienced? Our mind makes up for them with imagined experiences, whose possibility for infinite repetition gives us the ideal construction of an infinite sequence of numbers" [14].

The goal of the present paper is to document a new instance of a Sierpinski triangle that, to the best of our knowledge, has not been previously pointed, neither in print, nor on the web. The reason lies probably in the fact that it is contained in a frieze laying up at the height of three meters, high above the colonnade of a cloister inside the Basilica of St John in Lateran. The novelty, besides its very existence, is that the triangle is assembled in golden leaf. This material would be unfeasible in floors, and is usual in vertical furnishings. In this case, the golden frieze is situated in a cloister, a structure affording large surfaces, enclosed by walls and with no ceiling, thus benefitting from natural light. The geometrical cosmatesque motives are underlined by the glimmering.

The paper is organized as follows. In section 2 we describe precisely the golden leaf triangles of the cloister of St. John in Lateran, their placement and orientation. In section 3 we describe the frieze of St John, a review of its historical changes, the role of the families of *Marmorari*, and discuss the preserved medieval parts of the basilica. In section 4 we explain exactly what is mathematically a Sierpinski triangle, its several possible (equivalent) rather different constructions and its mathematical properties such as dimension and topological properties. We also go over its historical appearance and refinement as a precious mathematical tool, still much alive and in use.

2 The Sierpinski Triangles in the Cloister of St. John

The cloister is an enclosure; the empty space is used as reference for orientation. This means, for instance, that the south wall of the cloister is "southern" with respect to the inside empty space, so its surface is exposed actually facing geographical North and therefore it is the one never being directly struck by sunrays. In the following section we describe more precisely the difference in ornaments along the sides. The southern wall is today without elaborate golden ornaments; we think it could well have been always without much gold, as it still carries intact the green and red serpentine and porphyry plates, with no sign of further inlays. As reported in the following section, the southern wall, while poorer in geometric motives, is semantically full, by carrying an inscription important to the spirit of the place. The triangles we report, instead, are situated on the northern and eastern walls of the cloister. The most striking one, the 4-level triangle (Fig. 1), is rather central on the eastern wall, positioned on a large square pillar, its front advanced with respect to the rest of the frieze. It receives direct sunlight starting at mid-day and into the afternoon. Another triangle, iterated at 3 levels on a red porphyry *rota* is on the same wall. A third 3-level one can be seen on the northern wall, lying on a green serpentine *rota*. All in all, the decorations and inscriptions seem designed to be observed while in meditation.

All marble plates that presented golden inlays have become more fragile during the centuries, and have been more or less deteriorated by loosing some golden pieces and some of the small mosaic tesserae. However the antique placements in the mortar are still mostly visible today, even where some small pieces have fallen out. Similarly the beautiful and varied twisted columns have lost much of their original golden and vitreous paste inlays, but a careful observation allows the reconstruction the original decorative patterns. Medieval twisted columns are varied in decoration, and also in their structure, "twisted" well beyond what is usual in more recent centuries. A study of their twisting structure is reported in [15].

As noted, the 4-level triangle is in a prominent position: it fills the round field on one of the central pillars of the eastern side of the cloister, just on the left when entering the central space from that direction. It is an original decoration, since in that front no external masonry buttress to strengthen the cloister's vaulted structure was ever built. Unfortunately it lacks some portions; nevertheless what remains is sufficient to reconstruct without doubt the whole drawing. The base of the decoration is a white marble slab, probably from ancient Luni caves (*Lunense*, nowadays still used and known with the name of Carrara, in Tuscany, the region of origin). In the surfaces are carved the lodgments for mosaic bands wrapping the central circular part. The tesserae in these bands are disposed following two different regular patterns, based on 45° and 90° angles, more regular on straight segments, but fluently adapted to follow curved paths. These stripes are made of red and black vitreous paste, while the white *tesserae* are probably cut from thin slabs of *palombino* marble. In this coloured ensemble lay regularly placed golden tiles; examining where the thin golden foil and the upper glass leaf are lost, it is possible to state that the base of golden tesserae is red vitreous paste. The central round is obtained with a red porphyry *rota*, encircled with a single ring of triangular tiles of the four base colours: red, black, white and gold. Observing other parts of the frieze where the decoration is lost, we can establish that the porphyritic elements are usually thicker than the mosaic elements: the first have a deeper inlay and usually require a deeper carving in the marble plate. The porphyry round element is divided in three sectors and in the centre the triangular composition is lodged. The first and second level's triangles (the bigger ones) are still in porphyry, while the smaller ones (coming in two different sizes) are made in triangular tiles of vitreous paste. The gilded tesserae are all equilateral triangles of the same size, a golden powder filling all the regular spaces left in between the red elements. This choice is probably due to constructive reasons: it would have been quite difficult to cut the smaller tiles from thick porphyry slabs, but it may be noticed that the larger vitreous paste triangles are "special", being considerably bigger than the average of the other decorations. Moreover, all the central drawing is delineated using elements based on 60° angles.

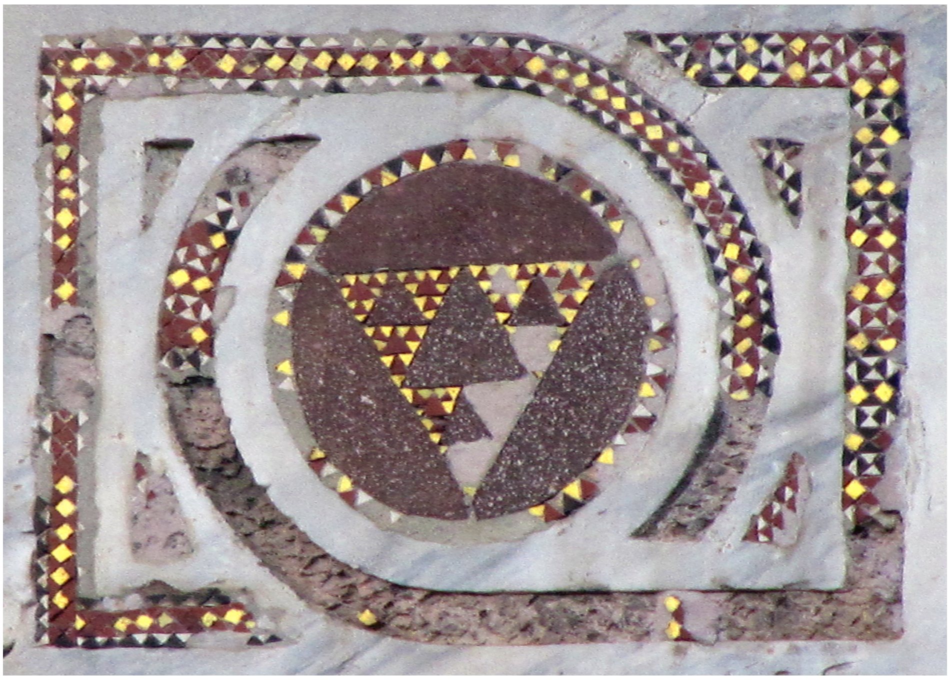


Fig. 1. The 4 level Sierpinski triangle in the cloister of St. John. Image processed by authors.

Only a careful and close observer will be able to notice the difference between the two red materials (porphyry and vitreous paste); the first sight shows only a red/gold decoration. The golden leaf, as said, is used in the smaller scale identical triangles, driving the onlooker's glance to this scale. The subtle shimmering of the gold further points to an even smaller scale, for the onlooker to consider. If an observer distinguishes the two materials, a single pattern proposed on two different scales can be seen: a bigger one (composed with the porphyry parts) is the central framework to which the smaller ones (the red vitreous parts) are attached. In turn, all the golden parts are attached and organized around the framework of all the red parts (triangles of 4 different sizes).

3 The Cloister of St John in Lateran and the Role of the *Marmorari*

The Basilica of St. John in Lateran, the Cathedral of Rome, was built by order of Emperor Constantine during the Pontificate of Sylvester starting on 314 (or 318), to be the residence of the Bishop of the City, on a piece of land that had been imperial property for centuries. The Lateran complex was for nine centuries the principal, but not the only, pontifical residency, long time before the construction of the Vatican palace; it was annexed to the Basilica "Mater and Caput", i.e. Mother and leader of all the churches of the City and of the world, as reported still today on the inscription on the XVIII century façade: *"Sacrosanta Lateranensis Ecclesia Omnium Urbis et Orbis Ecclesiarum Mater et Caput"* [4]. Well before year 1000, the symbols of ancient Rome were displayed in the *campus Lateranensis* modified in their meaning, to suggest that within a principle of continuity, the power of the Pope was the heir and continuer of ancient Rome imperial dignity. Starting on 1215, the *campus Lateranensis* was the seat of a Council called by Innocent III to establish the leadership of Rome over all the other Episcopal seats, and the supremacy of the Church over the Empire. From 1216, Honorius III begins the renovation of the Cloister annexed to the Basilica; the existing cloister, in fact, has replaced a more ancient one of the XII century built with stony columns bearing ionic capitals, and marble architrave.

Medieval Rome was a city rich in interrelated ferments [2]: the Pontificate aimed at re-establishing the ideals of the primitive Church architectonically symbolized by the basilicas structures; having established supremacy over the secular power, the Popes tend to impose a style that may symbolize the triumph of the roman Church and its Shepherd, who is not only the successor of Peter but the representative of Christ on earth. This trend corresponds to a cultural revolution that between the XI and the XIII century's aims at imposing extensive changes on pre-existing churches, concentrating particularly on the renovation of the interior while leaving the overall architectural plan almost unchanged. It is indeed a *Renovatio,* yet within continuity with the ancient spirit and the tradition and heritage of the magnificent roman past, presented and interpreted in a renovated paradigm.

This renovation makes extensive use of materials available in the ruins of the ancient monuments, as if they were huge quarries. The privilege to employ these ancient materials was assigned exclusively to a limited number of families that transmitted through generations their knowledge in employing the materials and handling the equipment, knowledge based on tradition coupled to technical skills. Approximately seven families were given the monopoly of these activities: they are collectively known as the *"Cosmati"* from the name of one of these clans operating in the second half of the XIII century. They are also referred to as *Marmorari romani*, (Roman marble workers).

The use of available ancient materials is economic advantage, and it also represents symbolically a direct and tangible connection with antiquity, thus glorifying the client while underlying the technical skill of the artist. Indeed these masters should be considered the true authors of the profound renovation of ancient roman churches: well aware of their function and achieving high social status, these marble workers will be, until the end of the XIII century, the only artists proudly signing their artifacts: sumptuous floorings with stony inlaid, carpets decorated by miniscule motifs, following rigorous geometrical patterns. The artists employed glassy melts and enamels, for coloured decorations not available in the ancient marbles, thus enriching the possible variants of the decorations and, in the footsteps of antique masterpieces, employed glassy paste coated by thin metal foils to obtain the shiny look of silver, and especially gold.

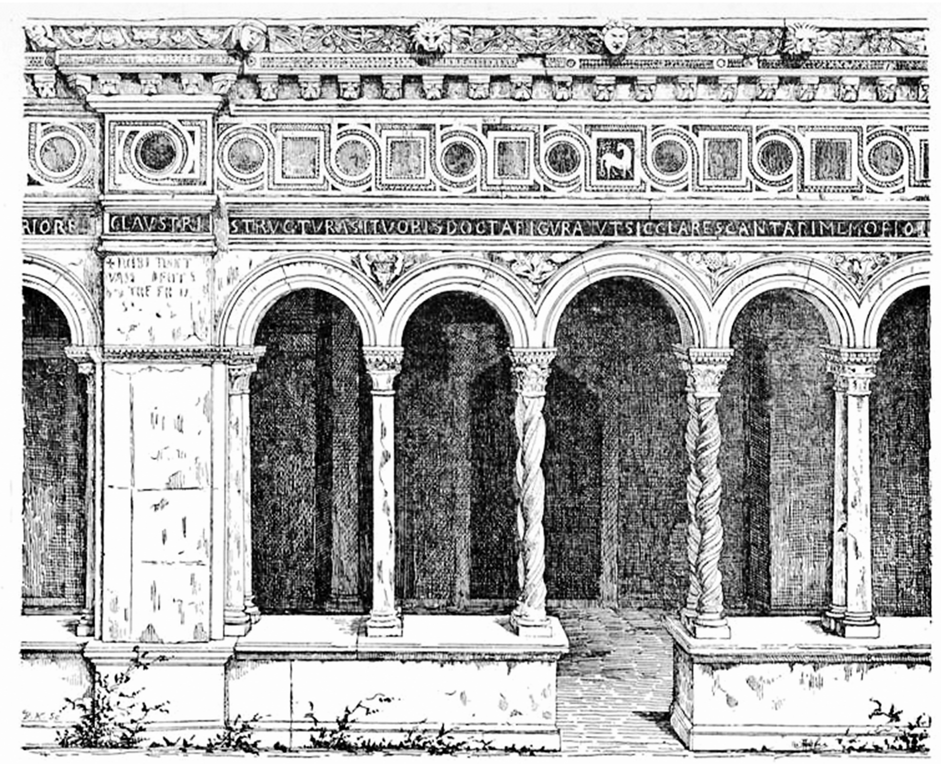
The construction of the Cloister of St. John began during the period of Pope Honorius III (1216-1227), and was completed before 1236; indeed an official deed of that year reveals the presence in the complex site of the Lateran of an old and a new Cloister. The structure and the decorations (sculptures and mosaics) in the "new" cloister represent masterpieces attributed to the family Vassalletto, a clan that along four generations of artists was active in Rome from the middle of the XII to the second half of the XIII century. The relatively short period employed to complete the Cloister's decorations is possibly witnessed by the fundamentally homogeneous stylistic trait of the whole ensemble, where some difference in style may be likely attributed to the one or the other of the Vassalletto family members.

The Cloister has a square plan, side of approximately 36 meters, and it displays similarities to the Cloister of the Basilica of St. Paul outside-the-walls; the deambulatory is covered by cross vaults built on corbels along the wall, and pillars twinned by columns from ancient sites of Egyptian basalt, *pavonazzetto* marble, granite, finished by ionic capitals. Each side of the porch is divided into five sectors; except in the northern side, nearer to the transept of the Basilica in the center of each side are open entrances to the central space. Flanking each entrance are lions and sphinxes, bearing the load of paired twisted columns. At the level of these column-bearing animals runs a molded basement, which acts as a base for the paired columns with Corinthian capitals. On these, settles a continual theory of little arches; in between the arches, vegetable and figurative motifs are carved, with references to fables and mythological tales, illustrating the struggle between good and evil,

Above the arches is articulated a tripartite trabeation; on the lower part, the architrave, a continuous molding runs and a long mosaic ribbon, partly decorated by geometrical drawings, embraces the four sides of the cloister. On the southern front, instead of the geometrical patterns, is a long inscription with white letters on a blue background that could be read from the northern side, privileged viewpoint: it is an allocution to the Canons, probably addressed by the Prior, which admonishes to follow the strict rule and recalls the main duties of the Canons: poverty, moral rigor, modesty. Comparing this inscription with those of the cloister of Saint Paul, a significant inversion is evident: in Saint Paul letters stand out dark on a golden background, figures on an abstract golden sky, much as in the Early Christian and byzantine tradition. In Saint John, gold assumes a material character and stands out clearly and radiantly only in the decorative patterns. In the cloister of St. John the severe words may appear in contrast with the splendor and the rich decorative apparatus, but the inscription itself offers an interpretation in which the aesthetics of the artists appears at the service of the moral rigor of the Basilica's Chapter. The clear planning of the architecture and its beauty are considered analogical figure of life according to the Rule. The shine and the glitter of the golden tiles of the mosaics refer to the splendor of the souls. The constructive act itself is exalted: the sanding of the stones leading to their perfect interlocking suggests the process of fortification of the spirit and of the perfect union of the members of the community. In the inscription neither a donor nor a lender of the work is mentioned. This instance confirmed the hypothesis that we can date the planning of the cloister to 1216, in conjunction with the founding of San John Hospital by Cardinal Giovanni Colonna.

The missing mention of donor and of the Abbot can be explained by the mutation of the address consequent the affirmation of a strict rule, thus even more surprising is the epigraph by which the Vassalletto signs the work ("*Nobiliter doctus hac Vassallectus in arte/ cum patre cepit opus quod solus perfecit ipse"*), settled just on the piers where the word "CLAUSTRI" shines, just under the phrase which indicates in the structure of the cloister a concrete example of a spiritual itinerary to follow. The Vassallettos, son and father (probably Petrus Vassallettus), sign the work in the name of tradition, an artistic continuity between generations, conveyance of techniques and knowledge that make the author "doctus". The two adjectives by which the epigraph begins are the sign of the social ascent of this class of artisans contractors in the first half of the XIII century in Rome; on many occasions the marble workers have the privilege of putting their own signature on their work adding attributes by which they proudly declare their artistic and technical abilities.

Over the inscription on the architrave the artists create the frieze in which circles and squares follow one another, surrounded by a marble ribbon that includes geometrical patterns made of mosaic with tesserae in stone, vitreous paste and enamel of four colors: white, red, gold, blue (dark blue almost black). They are arranged into regular drawings starting from tripartite and quadripartite geometric matrices. There is no rigid sequence of the colours: a changing between one and the other is a pleasant variety. Within the frieze, a few representational motifs rich in symbolic values are conserved; we recognize the Lamb crowned with a red nimbus, the bird in the cage, the knot of Salomon and other symbolic symmetrical interlacements, the Basilisk. The specific geometrical composition presented in the previous section is from one of the fields of the eastern side, located near the middle passage towards the central space of the porch, inside a circle with gold tiles and porphyritic parts. The decoration of the field is among the original ones, preserved (albeit with some lacunae) because they were not hidden by the masonry pillars that used to support the northern and western sides until the restoration at the end of the XIX century. The trabeation ends on the top with a molded and sculpted cornice of classical taste; on a recurring vegetable motif leonine protomes, faces, and masks line up.



Portique du cloître de la Basilique de Saint-Jean-de-Latran. Rome. — XIII^e siècle.
Inscription et signature de Vassaletus.

Fig. 2. Southern side of the cloister, view of the central sector. “VASS” on left pillar refers to Vassalletto ([7] p. 429).

Above all, the critics, starting from Giovannoni, [16] by analyzing stylistic features of these elements, have been able to discriminate the work of Vassalletto father and son: the first should be the author of the completion of the northern side and part of the eastern front, while the other two sides should be ascribed to a later period. The first phase is distinguished by the use of richer naturalistic shapes and marked references to antiquity even in the choice of the subjects. A sequence of big and different lions’ snouts lines up along the cornice, and in between is a cat’s head, a faun’s head that recalls a comedy mask, human’s smiling faces, naturalistic and with medieval hats. It is likely that they could be portraits of the same stonemasons. On the same façade we can see fantastic representation of dragons, sirens, devil masks and ingenious fusions of frontal and profile faces.

The palmette frieze on the northern side is deeply engraved, dug with the help of drills, especially when compared with that of the adjacent west side whose cut is more clumsy. The manufacture on the south side, despite the overall more simplified design, is good and the details imaginative.

Important differences have been also found in the make of the column-bearing lions flanking the passages towards the center of the cloister: those to the east, seated, are attributed to the Vassalletto father for the marked modeling; while those to the west, crushed by the overpowering weight and almost kept in trap, are more similar to other contemporary examples. Amazing is the pair of sphinxes that flanks the southern entrance, one bearded and the other beardless, deliberately deformed in their proportions to underline the demonic side, enigmatic in their expressions, one more frowning and the other almost smiling.

3.1 Bibliographic Note

Bibliography about the Basilica of St. John in Lateran is immense and, even limiting ourselves to texts that deepen the theme of the cloister, we would have to provide a very long list, beyond the scope of this paper. However, it's not possible to ignore the work of Peter Cornelius Claussen, the greatest scholar of the art of Roman marble makers, who reports and discusses critically the previous texts both on the global argument and specifically on the Cathedral of Rome [8, 9]. With regard to historical and artistic context, the reading of Richard Krautheimer [18] is still fundamental, while for the pleasantness of the text, for the breadth of horizons and the numerous references, even a non-specialist can enjoy the work of Jean-Claude Maire Viguer [20]. A theme as "Roman" as that of the Lateran cloister has been explored by the most attentive scholars of the Rome School of Architecture: very significant the study of Gustavo Giovannoni of 1908 [16]; even Leonardo Benevolo in 1954 presented the synthesis of a research on Cosmati's decorative geometric motifs [3], a subject that to date does not have an exhaustive treatment. Finally, a book on the medieval portico of St. John in Lateran is soon to be published for the Editions of the Vatican Museums (Anna Maria De Strobel ed.); it presents the result of a multidisciplinary investigation framing the theme of the portico on columns that adorned the front of the Basilica, decorated with a long cosmatesque mosaic frieze, a work of art that, both from a chronological point of view and as regards spatial proximity, constitute the immediate precedent of the cloister of St. John.

4 The Sierpinski Triangle, its Mathematical Values

Mathematicians consider Sierpinski's triangle a "study object", interesting *per se*: it has been invented, but still to this date it is studied in itself, as prototype and benchmark for many different theories, because it carries, with clarity, topological and dimensional deep properties; while it is easy to imagine and represent in drawing (obviously iterating only a finite number of times), what happens when calculating its "dimension" or "measure" is often counterintuitive, helping to question and probe the sense itself of these two terms.

Mathematically speaking, a Sierpinski triangle is a set obtained by inserting voids in a full planar region, in a recursive way, starting from a triangle. One of its possible constructions is the following (originally described by Sierpinski in his paper of 1915 [27]): let T be an equilateral triangle; mark the middle points of each edge; join these three points, thus obtaining 4 equilateral triangles (Fig. 3 left). Now think of keeping the triangles containing the vertexes A,B,C respectively, and discard the interior of the fourth triangle, the middle one. We are left with 3 triangles, arranged around a "void". Proceeding by iteration, we subdivide each of these 3 triangles, then we delete the inner ones, and we obtain 9 triangles (that is 3^2) as in Figure 3, right. This process leads to obtain, at the nth iteration, a set of exactly 3^n identical equilateral triangles, arranged in an exactly describable way around voids. Taking this procedure to the limit, mathematically, we then define the set S "Sierpinski triangle" to be the intersection of all these sets, that is $S = \cap_n S_n$. Loosely speaking it consists of all points of the plane that are not erased at any level of this recursive procedure, i.e. "what is left" after infinitely many iterations of erasing (smaller and smaller) central triangles.

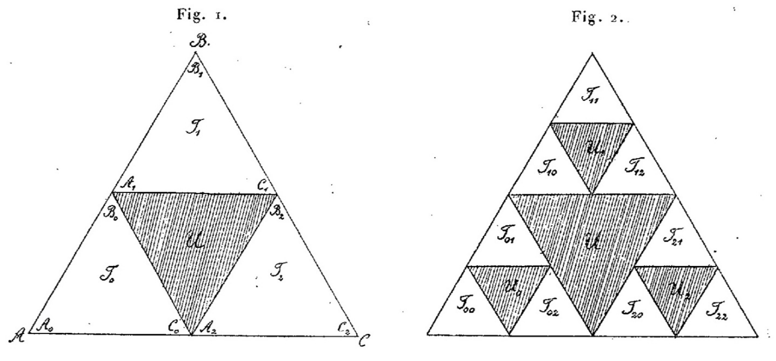


Fig. 3. The construction of the Sierpinski triangle, in the original article of 1915 [27]; level 1 iteration, left: level 2 iteration, right. In black the triangles that are erased at each step. In white what "is left".

The set S, thus defined, contains no isolated points, nor interior points: this is what makes it valuable to help new scientific thought and imagination. In other words, each point of S is an accumulation point for S (i.e. in every neighborhood of each point of *S,* however apparently small*,* there are infinitely many other points of *S*); moreover set S does not contain any open non-empty subset (as close as you want to any point of S there is a void, an area that has been discarded at some level). So it is a *continuum*, and nowhere *dense* in the plane, i.e. a set W is dense in X if for any point $P \in X$, there exists infinitely many points of W "arbitrarily close" to P. We will be clearer about the concept of continuum in next lines.

The Sierpinski triangle is also self-similar, since “it contains replicas of itself of many different sizes” ([1]); this property comes directly from its geometric construction.

At the time it was created, new theories about sets, numbers and topology were being invented, and carried new definitions of the concept of “dimension” and “measure”. At the end of the Nineteenth century a great step forward was made in the theory of sets and on the structure of numbers, regarding the accumulation of nearby numbers, and the new operation of the passage to the limit in a sequence of numbers. The main contributions came from the work of mathematicians such as Dedekind (1831-1916) and Cantor (1845-1918). We owe to them the definition of Real numbers, as we know them today (they gave two different but equivalent definitions [11, 21]).

The breakthrough contribution of Cantor is that he applies the concept of “equipotent” to sets consisting of infinite elements. He proves that when dealing with infinite sets, it may happen that a part of the whole set has the same *cardinality* as the whole set itself , i.e. one can define one-to-one correspondence between the whole and one of its strict parts. In the same period Dedekind gives the pertinent definition: a set S is infinite if it is equipotent to one of its proper subsets. Otherwise it is finite.

Mathematicians were understanding that infinite sets can be very different from each other: the cardinality of the *countable* (that of the Integers) is stricly less than that of the *continuous* (that of Real numbers). Objects of different *dimension* (a formalization of the concept of dimension still had not arrived), can be in a one-to-one correspondence: Cantor proved that a square has as many points as one of its sides (for an accessible discussion on the cardinality of infinite sets we refer to [19]).

Many mathematicians resumed Cantor's ideas on set theory, Sierpinski in particular began to teach these theories in his courses around 1908. Sierpinski was one of the major exponents of the Polish mathematical school, resting solidly on these new theories. He himself publishes works on the Hypothesis of the continuum and on transfinite numbers [11]. The Sierpinski triangle was one of the products blooming from these new concepts. Such sets are intersting and still used to this day, exactly because they counteract fallacious intuitions about “continuity”, forcing a more rigourous imagination. They actually were created in order to imagine new possibilities, and test new ideas and measurements.

The scope of the article of 1915 written by Sierpinski was to give an example of what he called a “Cantorian curve”, a very special new set. In fact Sierpinski shows how to approximate it by drawing actual level-n curves, starting from the geometrical construction of the level-n triangle: the curve *C* can be visualized as the limit of the sequence of the segments shown in Figure 4. It appears as a polygonal line, but from a mathematical point of view it is actually a curve. Furthermore he proves that each point (except for the three vertex of the original triangle) of the limit curve is a *branching* point; in other words, the curve crosses itself at every point [28].

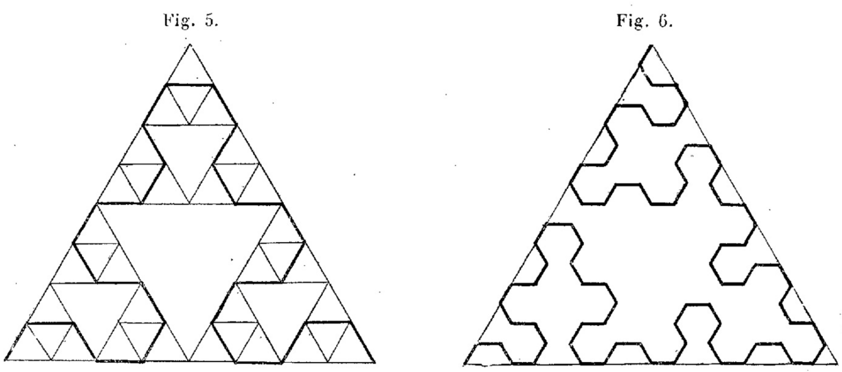


Fig. 4. The polygonal line on the right is the so called Sierpinski curve, from the original article of 1915 [27].

We have just described what we call a top-down procedure: we started from a “full” object, the equilateral triangle, and by inserting voids, we arrived to the Sierpinski triangle. The same set can be defined with a down-top process, so adding points, following a dynamical system, called "Chaos game"[1]. Let A, B, C be three points on the plane, like the vertex of an equilateral triangle. Start on a point, P, randomly chosen in the triangle. Move P half of the path that separates P from one of the three vertexes, choosing randomly the vertex. Iterate, at each step moving towards one of the vertexes, choosen randomly, and drawing the arrival point. After drawing many points, these will accumulate on a Sierpinsky triangle(Fig. 5).

The interest of these down-top procedures is that the Sierpinski triangle is attained as “attractor” of a dynamical system (the Chaos game), and that by iterating the Chaos game one sees point appearing one after the other, while an overall organization emerge.

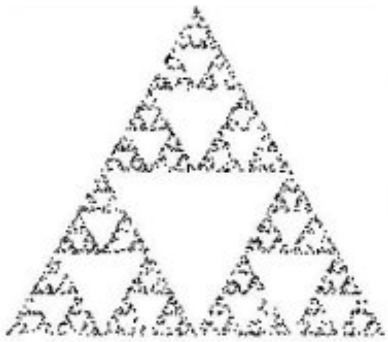


Fig. 5. The "self assembling" of the points obtained by the iteration of the Chaos Game. Image by the authors.

This point of view helped, for instance, the study of "self assembling structures" [26, 29] in chemistry.

4.1 The Sierpinski Triangle has a non-integer number of "dimensions" (a fractal).

As it usually happens in mathematics, new theories originate new objects and examples, which, in turn, originate new tools. In this case, sets as the Sierpinski triangle, contributed to new definitions of "dimension" and of "measure". When we say just "dimension" of an object, we usually mean the topological dimension, which can be thought of as the minimum number of parameters needed to define the coordinate of a point belonging to the object, in a Cartesian system [6]. A curve has dimension 1, since its description can be formulated so its points depend on a single parameter. The surface of a sphere has dimension 2, since a point on a sphere needs at least two parameters to be described…

When dealing with a set such as the Sierpinski triangle, one has to come up with new ideas, encompassing the previous known cases, and allowing new sets. The set S has at the same time the characteristic of a line (whose dimension is 1), but necessarily needs an ambient space of at least dimension 2 to be described. The whole idea of "number of parameters" seems inadequate.

New ideas about "dimensions" came up, dealing with how much a set fills its ambient space. A cube fills entirely a portion of the 3d space, and therefore is of dimension 3. We will use in the following the so-called "box-counting" dimension, particularly suitable for sets defined in a recursive way (see [25] and the reference therein). This dimension is particularly suitable for computer experiments.

These dimensions are usually computed by covering the object with a grid of squares (more in general n-dimensional neighborhoods, depending on the dimension of the ambient space), and then computing the limit defined in the following lines. Suppose a set A is contained in the plane; to describe it, first cover the set with a grid of squares, having side length ε. The number N_ε indicates the minimum number of squares of side ε needed to cover the object. So ε is the "resolution" of the description of the set. If we lower ε, enhancing the resolution, we will need a larger number of squares. The "box counting dimension" computes the growth of information needed in passing to higher resolution, and thus studying smaller and smaller details. It is defined by the following limit (provided it exists):

$$\mathrm{Dim}(A) = -\lim\ (\ln (N_\varepsilon)/\ln(\varepsilon)), \quad \text{as } \varepsilon \to 0.$$

If the choice of the term "dimension" is correct, this should show familiar integer numbers in familiar cases, encompassing what we knew, while allowing for more complicated sets. Let us show as an example how to compute the box-counting dimension for a finite curve ([24]). Let the curve have length L; if the squares have side ε, the minimum number needed will be approximately $N_\varepsilon \approx L/\varepsilon = L\varepsilon^{-1}$. So as $\varepsilon \to 0$, $\ln(N_\varepsilon) \approx \ln(L) - \ln(\varepsilon)$. Now calculating the limit gives 1, which is the result we expected. Similarly, for a plane figure, such as, say, a square, we expect $N_\varepsilon \approx \varepsilon^{-2}$. Therefore a dimension d is the rate of growth of $N_\varepsilon \approx \varepsilon^{-d}$.

Number d could well be non-integer. If this is the case, it means that the set A does not fill entirely an area of the plane, but it needs a 2 dimensional ambient space to be defined and live. The calculation of the box-counting dimension for a Sierpinski triangle can be found in [10] and gives the result $d = \ln 3/\ln 2$. Furthermore, the Sierpinski triangle has zero area: this can be computed calculating the area of the 3n triangles of level n iteration, and then passing to the limit on n.

The Sierpinski triangle keeps gathering interest; to give an example, at the end of the 1980's physicists had the intuition that some unusual diffusion appearing in phenomena like percolation, observed in some particular materials, could be related to the *ramification* at all scales inside the material itself. The "ramifications" (or branching points) from a topological point of view, are much the same as those described by Sierpinski in his seminal paper of 1915. Mathematicians in turn, at the end of 1980's developed new tools to construct rigorously a diffusion process on structures showing ramifications derived from the Sierpinski triangle (see [23] and the references therein). This is new to the idea that a mathematical model of "diffusion" needs to be carried over smoothly continuous sets, in order to be able to use tools of infinitesimal calculus and analysis. As the molecular structure of matter comes more and more under the study, new mathematical methods are developed, dealing with such new sets in the small scale.

5 Conclusions

In this paper we reported about some Sierpinski triangles in golden leaf present in the medieval frieze of the Cloister of St. John in Lateran, Rome. We make the point that this composition has gone untouched for centuries. We also point to the importance of the family of Marmorari specifically linked with the whole Cloister, and with this composition.

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