Find an equation of the hyperbola that satisfies the conditions in each of Exercises 11–18.

11 Center (0, 0); vertex (0, 12); focus (0, 13)

12 Center (0, 0); vertex (4, 0); focus (5, 0)

13 Vertices $(-4, 3)$ and (4, 3); focus (5, 3)

14 Vertex (0, 0); foci $(0, -2)$ and (0, 8)

15 Vertices $(1, -3)$ and (1, 3); contains (0, 5)

16 Vertices (0, 2) and (6, 2); contains $(-2, 0)$

17 Center (0, 0); vertex (0, 4); contains (1, 6)

18 Center (0, 0); vertex (3, 0); contains (4, 5)

19 Find equations of all hyperbolas that have focus at the origin, vertex (2, 0), and eccentricity 4.

20 Find equations of all ellipses that have focus at the origin, vertex (0, 3), and eccentricity 3.

21 Describe the set of all (x, y) that satisfy the condition that the difference of the distances between (x, y) and each of the points $(-c, 0)$ and $(c, 0)$ is $2c$, $c > 0$.

22 Describe the set of all (x, y) that satisfy the condition that the difference of the distances between (x, y) and each of the points $(-c, 0)$ and $(c, 0)$ is $2d$, $d > c > 0$.

23 Find an equation of all (x, y) that satisfy the condition that the distance between (x, y) and the origin is twice the distance of (x, y) from the line $x = -3$.

24 Find an equation of all (x, y) that satisfy the condition that the distance between (x, y) and the origin is e times the distance of (x, y) from the line $x = d$, $e > 1$, $d \neq 0$.

25 The hyperbola $16x^2 - 9y^2 = 144$ is sketched below.

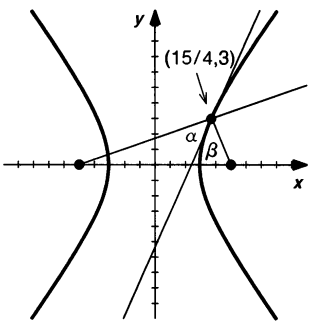
(a) Find the slope of the line tangent to the hyperbola at the point (15/4, 3).

(b) Find the foci of the hyperbola.

(c) Find the slopes of the lines through (15/4, 3) and each focus.

(d) Find $\tan\alpha$ and $\tan\beta$. [Note that the tangent of the angle between lines that have slopes m_1 and m_2 is $(m_2 - m_1)/(1 + m_1m_2)$.]

(e) Find a relation between the angles α and β. (This verifies the reflection property of this hyperbola at this point on the hyperbola.)



26 The hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, $a > 0$, $b > 0$, is sketched below.

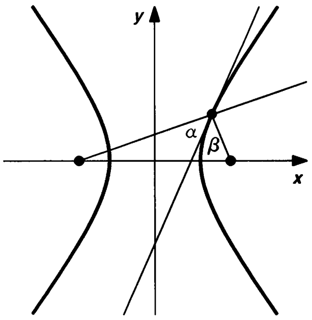
(a) Find the slopes of the lines tangent to the hyperbola at a point (x_0, y_0) on the hyperbola, $y_0 \neq 0$.

(b) Find the foci of the hyperbola.

(c) Find the slopes of the lines through (x_0, y_0) and each focus.

(d) Find $\tan\alpha$ and $\tan\beta$. [Note that the tangent of the angle between lines that have slopes m_1 and m_2 is $(m_2 - m_1)/(1 + m_1m_2)$.]

(e) Find a relation between the angles α and β. (This verifies the reflection property for hyperbolas.)



27 Use Figure 11.3.2 to verify that the equation $d_2 - d_1 = \pm 2a$ leads to the standard form of the equation of the hyperbola.

11.4 TRANSLATION AND ROTATION

We will see two different ways to change from a given coordinate system to a new coordinate system. The new coordinates will be used to simplify and analyze equations given in terms of the original coordinates.

Translation

The relation between the coordinates of a point with respect to a rectangular coordinate system (x, y) and a rectangular coordinate system

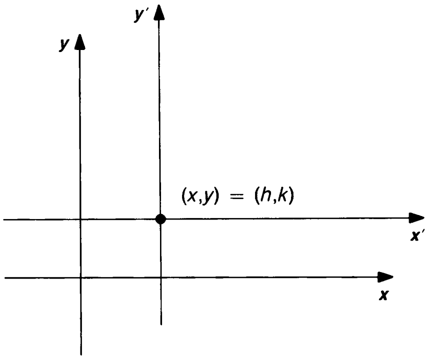


FIGURE 11.4.1

(x', y') obtained by **translation** of the origin to the point $(x, y) = (h, k)$ is given by the equations

$$\begin{cases} x = x' + h, \\ y = y' + k. \end{cases}$$

This translation is illustrated in Figure 11.4.1. Note that the translated x'-axis is parallel to the original x-axis; the translated y'-axis is parallel to the original y-axis. From the equation $x = x' + h$, we see that $x' = 0$ corresponds to $x = h$. The equation $y = y' + k$ tells us that $y' = 0$ corresponds to $y = k$. That is, the origin of the translated system, $(x', y') = (0, 0)$, corresponds to the point $(x, y) = (h, k)$.

The equations of translation,

$$\begin{cases} x = x' + h, \\ y = y' + k, \end{cases}$$

give the original (x, y) coordinates in terms of the new coordinates (x', y'). This form is convenient for substitution in an equation that involves x and y in order to obtain an equation in x' and y'. The translated coordinates (x', y') can be expressed in terms of the original coordinates by the equivalent system of equations

$$\begin{cases} x' = x - h, \\ y' = y - k. \end{cases}$$

■ **EXAMPLE 1**

Express the equation $3x^2 - 12x - 8y = 0$ in terms of the (x', y') coordinates given by the translation

$$\begin{cases} x = x' + 2, \\ y = y' - \dfrac{3}{2}. \end{cases}$$

Sketch the graph and both sets of coordinate axes.

SOLUTION

Substitution of the equations of translation gives

$$3x^2 - 12x - 8y = 0,$$

$$3(x' + 2)^2 - 12(x' + 2) - 8\left(y' - \frac{3}{2}\right) = 0,$$

$$3x'^2 + 12x' + 12 - 12x' - 24 - 8y' + 12 = 0,$$

$$3x'^2 - 8y' = 0.$$

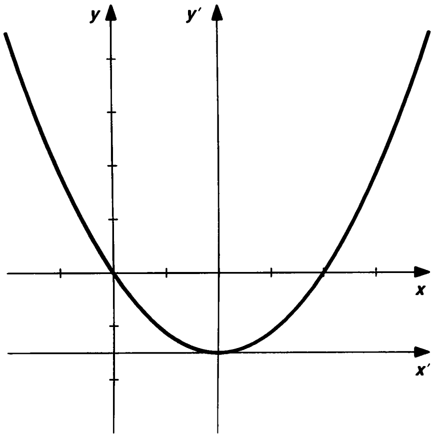


FIGURE 11.4.2

We see that the graph is a parabola with vertex at $(x', y') = (0, 0)$. The equations of translation tell us that $(x', y') = (0, 0)$ corresponds to $(x, y) = (2, -3/2)$. The (x', y') coordinate axes have been drawn with origin at the point $(x, y) = (2, -3/2)$ and the graph is sketched in Figure 11.4.2. ■

■ **EXAMPLE 2**

Find the equations of translation that translate the origin to the center of the ellipse $x^2 - 4x + 4y^2 = 0$. Express the equation in terms of the translated coordinates (x', y'). Sketch the graph and both sets of coordinate axes.

SOLUTION

The center of the ellipse is found by completing the square and writing the equation in standard form. We have

$$x^2 - 4x + [4] + 4y^2 = [4],$$

$$(x - 2)^2 + 4y^2 = 4,$$

$$\frac{(x-2)^2}{2^2} + \frac{y^2}{1^2} = 1.$$

We see that the ellipse has center $(h, k) = (2, 0)$. The desired equations of translation are then

$$\begin{cases} x = x' + 2, \\ y = y'. \end{cases}$$

[It is a good idea to check that $(x', y') = (0, 0)$ corresponds to the proper (x, y) coordinates, $(x, y) = (2, 0)$ in this case.] In terms of the (x', y') coordinates, the equation is

$$\frac{x'^2}{2^2} + \frac{y'^2}{1^2} = 1.$$

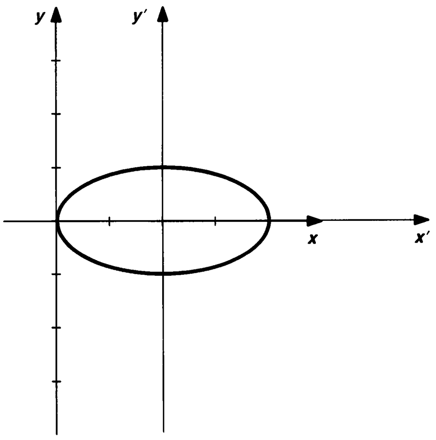


FIGURE 11.4.3

The (x', y') coordinate axes have been drawn with origin at $(x, y) = (2, 0)$ and the graph sketched in Figure 11.4.3. ■

Rotation

We want to establish a new coordinate system by rotating the x and y axes about the origin by an angle θ. This is illustrated in Figure 11.4.4.

A point P in the plane has coordinates (x, y) in the original coordinate system, and coordinates (x', y') in the new, rotated system. We want to determine the relation between the two coordinate systems. From Figure 11.4.4, we have

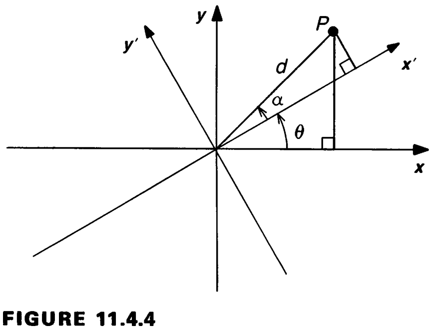


FIGURE 11.4.4

$$x' = d \cos \alpha \quad \text{and} \quad y' = d \sin \alpha.$$

Also,

$$x = d\cos(\alpha + \theta)$$
$$= d[\cos\alpha\cos\theta - \sin\alpha\sin\theta]$$
$$= x'\cos\theta - y'\sin\theta,$$

and

$$y = d\sin(\alpha + \theta)$$
$$= d[\cos\alpha\sin\theta + \sin\alpha\cos\theta]$$
$$= x'\sin\theta + y'\cos\theta.$$

Summarizing, we have

$$\begin{cases} \boldsymbol{x = x'\cos\theta - y'\sin\theta,} \\ \boldsymbol{y = x'\sin\theta + y'\cos\theta.} \end{cases}$$

Substitution of the above values of x and y in an equation gives an equivalent equation in terms of the new coordinates that are obtained by rotation of the (x, y) axes by an angle of θ.

The equations of rotation can be solved for x' and y' in terms of x and y. We obtain

$$\begin{cases} \boldsymbol{x' = x\cos\theta + y\sin\theta,} \\ \boldsymbol{y' = -x\sin\theta + y\cos\theta} \end{cases}$$

[Note that we can interpret these equations as corresponding to a rotation of the (x', y') axes by an angle of $-\theta$.]

■ **EXAMPLE 3**

Express the equation $2x - 3y + 6 = 0$ in terms of the (x', y') coordinates given by a rotation of $\theta = \tan^{-1}(2/3)$. Sketch the graph and both sets of coordinate axes.

SOLUTION

A representative sketch of the angle $\theta = \tan^{-1}(2/3)$ is given in Figure 11.4.5. We see that $\sin\theta = 2/\sqrt{13}$ and $\cos\theta = 3/\sqrt{13}$. Substitution into the equations of rotation then gives

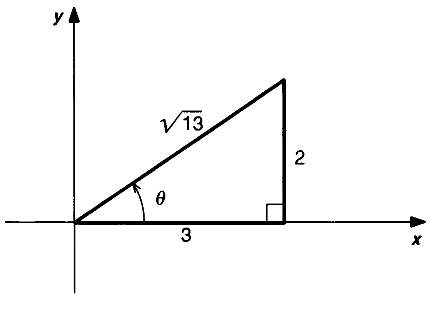


FIGURE 11.4.5

$$x = x'\cos\theta - y'\sin\theta = \frac{3x' - 2y'}{\sqrt{13}},$$

$$y = x'\sin\theta + y'\cos\theta = \frac{2x' + 3y'}{\sqrt{13}}.$$

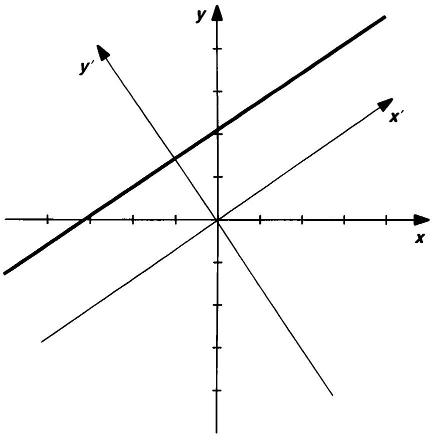


FIGURE 11.4.6

We then have

$$2x - 3y + 6 = 0,$$

$$2\left(\frac{3x' - 2y'}{\sqrt{13}}\right) - 3\left(\frac{2x' + 3y'}{\sqrt{13}}\right) + 6 = 0,$$

$$6x' - 4y' - 6x' - 9y' + 6\sqrt{13} = 0,$$

$$y' = \frac{6\sqrt{13}}{13}.$$

The graph and both sets of axes are sketched in Figure 11.4.6. Note that the line is parallel to the x'-axis. This is true because the angle of rotation θ is equal to the angle between the line and the x-axis. ■

■ **EXAMPLE 4**

Express the equation $xy - x + y = 0$ in terms of coordinates (x', y') that correspond to a rotation of θ. Find $0 < \theta < \pi/2$ such that the new equation contains no $x'y'$ term. Sketch the graph and both sets of axes.

SOLUTION

Substitution gives

$$xy - x + y = 0,$$

$$(x' \cos\theta - y' \sin\theta)(x' \sin\theta + y' \cos\theta)$$
$$-(x' \cos\theta - y' \sin\theta) + (x' \sin\theta + y' \cos\theta) = 0,$$

$$(\cos\theta \sin\theta)x'^2 + (\cos^2\theta - \sin^2\theta)x'y' - (\sin\theta\cos\theta)y'^2$$
$$+ (\sin\theta - \cos\theta)x' + (\sin\theta + \cos\theta)y' = 0.$$

We see that the new equation will contain no $x'y'$ term if θ is chosen so that

$$\cos^2\theta - \sin^2\theta = 0.$$

This gives $\tan\theta = \pm 1$. This equation and the condition that $0 < \theta < \pi/2$ are satisfied by the angle $\theta = \pi/4$. Since $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, substitution of $\theta = \pi/4$ into the above equation in (x', y') gives

$$\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right)x'^2 + \left(\frac{1}{2} - \frac{1}{2}\right)x'y' - \left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right)y'^2 + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)x'$$
$$+ \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)y' = 0,$$

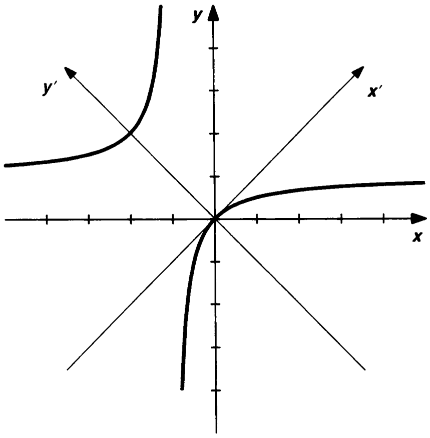


FIGURE 11.4.7

$$x'^2 - y'^2 + \frac{4}{\sqrt{2}}y' = 0,$$

$$x'^2 - y'^2 + 2\sqrt{2}y' = 0.$$

Completing the square to write this equation in standard form, we obtain

$$x'^2 - (y'^2 - 2\sqrt{2}y' + [2]) = -([2]),$$

$$x'^2 - (y' - \sqrt{2})^2 = -2,$$

$$-\frac{x'^2}{(\sqrt{2})^2} + \frac{(y' - \sqrt{2})^2}{(\sqrt{2})^2} = 1.$$

We see that this is the equation of a hyperbola with center $(x', y') = (0, \sqrt{2})$. The vertices are $(x', y') = (0, 0)$ and $(x', y') = (0, 2\sqrt{2})$. The graph and both sets of axes are sketched in Figure 11.4.7. ■

■ **EXAMPLE 5**

Find the (x, y) coordinates of the center, vertices, and foci of the hyperbola $xy - x + y = 0$ of Example 4. Find the equations of the asymptotes in terms of (x, y).

SOLUTION

From the equation obtained in Example 4,

$$-\frac{x'^2}{(\sqrt{2})^2} + \frac{(y' - \sqrt{2})^2}{(\sqrt{2})^2} = 1,$$

we see that the center is $(x', y') = (0, \sqrt{2})$. Since the angle of rotation is $\theta = \pi/4$, the corresponding (x, y) coordinates are

$$x = x' \cos\theta - y' \sin\theta = (0)\left(\frac{1}{\sqrt{2}}\right) - (\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) = -1,$$

$$y = x' \sin\theta + y' \cos\theta = (0)\left(\frac{1}{\sqrt{2}}\right) + (\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) = 1.$$

The center is $(x, y) = (-1, 1)$. Similarly, the vertices $(x', y') = (0, 0)$ and $(x', y') = (0, 2\sqrt{2})$ correspond to $(x, y) = (0, 0)$ and $(x, y) = (-2, 2)$, respectively.

We need the (x', y') coordinates of the foci. From the x', y' equation of the hyperbola, we see that $c^2 = a^2 + b^2 = 2 + 2 = 4$, so $c = 2$. The foci are $(x', y') = (0, \sqrt{2} - 2)$ and $(x', y') = (0, \sqrt{2} + 2)$. These correspond to $(x, y) = (\sqrt{2} - 1, -\sqrt{2} + 1)$ and $(x, y) = (-\sqrt{2} - 1, \sqrt{2} + 1)$, respectively.

The equations of the asymptotes are

$$-\frac{x'^2}{(\sqrt{2})^2}+\frac{(y'-\sqrt{2})^2}{(\sqrt{2})^2}=0,$$

or $y' = x' + \sqrt{2}$ and $y' = -x' + \sqrt{2}$. Using the equations $x' = x\cos\theta + y\sin\theta$, $y' = -x\sin\theta + y\cos\theta$, $\theta = \pi/4$, and substituting, we have

$$y' = x' + \sqrt{2},$$

$$(-x\sin\theta + y\cos\theta) = (x\cos\theta + y\sin\theta) + \sqrt{2},$$

$$-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}=\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}+\sqrt{2},$$

$$-\frac{2x}{\sqrt{2}}=\sqrt{2},$$

$$x = -1.$$

Similarly, the equation $y' = -x' + \sqrt{2}$ can be shown to have (x, y) equation $y = 1$. ■

The equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

(A, B, C not all three zero) is called the **general second-degree equation.** Rotation of axes by an angle θ transforms this equation to the equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0,$$

where

$$A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta,$$

$$B' = B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta,$$

$$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta,$$

$$D' = D\cos\theta + E\sin\theta,$$

$$E' = -D\sin\theta + E\cos\theta,$$

$$F' = F.$$

In particular, we see that we can obtain an equation that has no $x'y'$ term if we choose θ such that

$$B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta = 0.$$

We can use trigonometric identities for double angles to solve this equation for θ. That is,

$$B \cos 2\theta + (C - A) \sin 2\theta = 0,$$

$$\cot 2\theta = \frac{A - C}{B}.$$

If $B \neq 0$, this equation will be satisfied by an angle $0 < \theta < \pi/2$. If $B = 0$, the original equation has no xy term.

It is easy to analyze the equation in terms of the (x', y') coordinates, if that equation contains no $x'y'$ term. That is, $A'x'^2 + B'x'y' + C'y^2 + D'x' + E'y' + F' = 0$ is an ellipse (including the degenerate case of a single point) whenever $B' = 0$ and A' and C' have the same sign. In this case $B'^2 - 4A'C' < 0$. The graph is a hyperbola (or the degenerate case of intersecting lines) whenever $B' = 0$ and A' and C' have opposite sign, so $B'^2 - 4A'C' > 0$. The equation will be a parabola (or the degenerate case of a line) whenever $B' = 0$ and one of A' and C' is zero, so $B'^2 - 4A'C' = 0$.

The expression

$$B^2 - 4AC$$

is called the **discriminant** of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

It can be shown that the equation

$$A'x'^2 + B'x'y' + C'y^2 + D'x' + E'y' + F' = 0$$

that is obtained by any rotation satisfies

$$B'^2 - 4A'C' = B^2 - 4AC.$$

We can then conclude the following **discriminant test** from the discussion above.

If

$$\boldsymbol{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,}$$

then

$B^2 - 4AC < 0$ implies the graph is an ellipse,

$B^2 - 4AC > 0$ implies the graph is a hyperbola, and

$B^2 - 4AC = 0$ implies the graph is a parabola.

(Each case includes degenerate cases.)

EXERCISES 11.4

Express the equations in Exercises 1-4 in terms of the (x', y') coordinates of the given translations. Sketch the graphs and both sets of axes.

1 $x = y^2 - 1$; $x = x' - 1$, $y = y'$

2 $x + y^2 + 1 = 0$; $x = x' - 1$, $y = y'$

3 $y = 2x - x^2$; $x = x' + 1$, $y = y' + 1$

4 $y = x^2 + 4x$; $x = x' - 2$, $y = y' - 4$

Find the equations of translation that transform the origin to the center of the graphs of Exercises 5-8. Sketch the graphs and both sets of axes.

5 $x^2 - 4x + 4y^2 - 8y + 7 = 0$

6 $x^2 + 9y^2 - 18y + 8 = 0$

7 $x^2 - y^2 + 2y = 0$

8 $4x^2 - 12x - 9y^2 + 18y + 9 = 0$

Express the equations in Exercises 9-12 in terms of the (x', y') coordinates obtained by rotation by the given angle θ. Sketch the graphs and both sets of axes.

9 $x - y + 2 = 0$; $\theta = \pi/4$

10 $\sqrt{3}x + y = 3$; $\theta = \pi/6$

11 $2x + y = 2$; $\theta = \tan^{-1}(1/2)$

12 $3x - 4y + 12 = 0$; $\theta = \tan^{-1}(3/4)$

In Exercises 13-18, find $0 < \theta < \pi/2$ so rotation by θ gives an equation with no $x'y'$ term. Express the equations in terms of the (x', y') coordinates obtained by that rotation. Sketch the graphs and both sets of axes.

13 $xy + 1 = 0$ **14** $xy = x + 1$

15 $x^2 - 2xy + y^2 - \sqrt{2}x - \sqrt{2}y = 0$

16 $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y = 0$

17 $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$

18 $31x^2 + 10\sqrt{3}xy + 21y^2 = 144$

19 Find $-\pi/2 < \theta < \pi/2$ so rotation by θ changes the equation $4x + 3y = 12$ into an equation with no x' term. Express the equation in terms of the (x', y') coordinates obtained by that rotation. Sketch the graph and both sets of axes.

20 Find $-\pi/2 < \theta < \pi/2$ so rotation by θ changes the equation $ax + by = c$, $b \neq 0$, into an equation with no x' term. Express the equation in terms of the (x', y') coordinates obtained by that rotation.

21 Verify that rotation by any angle θ changes the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

to an equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

with $B'^2 - 4A'C' = B^2 - 4AC$.

22 Use the discriminant test to identify the graphs in Exercises 13-18.

REVIEW EXERCISES

1 Find an equation of the ellipse with center (4, 0), one focus (0, 0), and contains (4, 3).

2 Find an equation of the ellipse with foci (−3, 0) and (3, 0), and contains (0, 2).

3 Find an equation of the hyperbola with foci (−1, 0) and (9, 0), and eccentricity 1.25.

4 Find an equation of the hyperbola with vertices (0, 0) and (−2, 0), and one focus (1, 0).

5 Find an equation of the parabola with directrix $x = -2$ and focus (2, 0).

6 Find an equation of the parabola with vertical axis that has vertex (0, −2) and contains the point (−1, −1).

Sketch the graphs of the equations in Exercises 7-12.

7 $x^2 = 16 - 4y^2$ **8** $4x^2 + y^2 - 4y = 0$

9 $y^2 = 2 + x$ **10** $x^2 = 8 - 8y$

11 $\frac{y^2}{4} - \frac{x^2}{1} = 1$

12 $4x^2 - y^2 - 8x = 0$

In Exercises 13–14, find $-\pi/2 < \theta < \pi/2$ so the equations

$$\begin{cases} x = x' \cos \theta - y' \sin \theta, \\ y = x' \sin \theta + y' \cos \theta, \end{cases}$$

transform the given equation to an equation in (x', y') with no x' term. Express the equation in terms of the corresponding (x', y') coordinates. Sketch both sets of axes and the graph of the equation.

13 $y = \sqrt{3}(x + 1)$

14 $x + y = 2$

In Exercises 15–16, find $0 < \theta < \pi/2$ so the equations

$$\begin{cases} x = x' \cos \theta - y' \sin \theta, \\ y = x' \sin \theta + y' \cos \theta. \end{cases}$$

transform the given equation to an equation in (x', y') with no $x'y'$ term. Express the equation in terms of the corresponding (x', y') coordinates. Sketch both sets of axes and the graph of the equation.

15 $xy = y + 1$

16 $2x^2 + \sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y = 0$

17 Find the coordinates of the vertices of the ellipse $3x^2 - 2xy + 3y^2 = 4$.

18 Find the coordinates of the vertices of the hyperbola $3x^2 + 10xy + 3y^2 = 8$.